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Exponential Stability of Gyroscopically Stabilized Multi-Degree-of-Freedom Unstable Potential Systems Using Linear Damping

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This paper shows that N-degree-of-freedom generic unstable potential systems, whose potential matrices have an even number of negative eigenvalues and which are gyroscopically stabilized, can always be made exponentially stable through the use of an uncountably infinite number of indefinite damping matrices. A step-by-step methodology is provided for gyroscopic stabilization of such unstable potential systems that guarantees their exponential stability through the simultaneous use of positive and negative velocity feedback. Dissipative damping and positive velocity feedback are shown to constructively cooperate to bring about such stability. In contrast to the well-known Kelvin–Tait–Chetaev paradigm, which states that gyroscopically stabilized unstable potential systems are always made unstable in the presence of the slightest dissipation of energy, the paper points to a new paradigm, which states that the proper simultaneous dissipation and infusion of energy into generic gyroscopically stabilized unstable potential systems guarantees their exponential stability. What is meant by "generic" is explicitly stated, and such generic systems are shown to include most real-life N-degree-of-freedom unstable potential systems.

Nomenclature

a_i	=	distinct numbers in a set of $2n$ real numbers
c_i	=	positive diagonal elements of D_s^{Diss}
D, $\tilde{D}$, D_i^I	=	indefinite damping matrices
D_s^{Diss}	=	positive definite damping matrix
d_i, $\bar{d}_i$	=	positive diagonal element of submatrix D_i^I
G, $\tilde{G}$, $\hat{G}$, $\bar{G}G_i$, $\bar{G}_i$	=	skew-symmetric matrices
g_i, $\bar{g}_i$	=	element of skew-symmetric submatrices G_i, $\bar{G}_i$
I_r	=	number of pairs with identical negative entries in a pairing
$\tilde{K}$, $\hat{K}$	=	unstable potential matrix
K_{p-}, K_{p+}	=	row vector of negative and positive eigenvalues of $\hat{K}$
$\bar{K}_{p-}$	=	row vector of distinct negative eigenvalues of $\hat{K}$
$\tilde{M}$	=	positive definite mass matrix
m_i, m_l	=	multiplicity of negative eigenvalues of $\hat{K}$
N	=	degrees of freedom of unstable potential system
$2n$	=	number of negative eigenvalues of $\tilde{K}$, $\hat{K}$
q, x, y	=	n-by-1 column vectors
r	=	$m_l - n > 0$, minimum number of pairs that have identical negative entries in any pairing
r_i, $\bar{r}_i$	=	ordered quadruple of gyroscopic potential subsystem
s_i, $\bar{s}_i$	=	ordered sextuple of damped gyroscopic potential subsystem
T, W	=	real N-by-N matrices
u_i	=	2-by-1 column vector
$-\alpha$, $-\alpha_i$, $-\bar{\alpha}_i$	=	negative elements of indefinite damping submatrices
δ, δ_i, $\bar{\delta}_i$	=	nonzero real number
Λ, Λ_i, Λ_s	=	diagonal matrices
$-\lambda_i$	=	negative eigenvalue of $\hat{K}$

I. Introduction

IT HAS long been known that unstable linear potential systems that have an even number of degrees-of-freedom can be stabilized using gyroscopic forces. The introduction of dissipative damping in such systems, however, makes them unstable. This seminal result obtained by the joint work of Peter Tait and Lord Kelvin began in 1861 and culminated in 1867 [1]. Their book, titled *A Treatise on Natural Philosophy*, develops this idea from physical considerations, and their result was mathematically proved in the 1950s by the mathematical physicist Chetaev [2]. Though nonintuitive (because linear dissipative damping extracts energy from systems and in most systems therefore aids their stability), this result, which is known today as the celebrated Kelvin–Tait–Chetaev (KTC) theorem, is an important cornerstone of the theory of linear stability [3,4]. It is of great practical value because it correctly predicts the behavior of gyroscopically stabilized unstable systems in which the damping matrix is positive definite. It has been handed down to the scientific, mathematical, and engineering communities since more than a hundred and fifty years and has become one of the important paradigms in stability theory.

The KTC theorem deals solely with systems subjected to three qualitatively different forces: conservative positional forces that make the system unstable, called unstable potential systems for short; gyroscopic forces; and linear-in-velocity damping forces. This paper too deals solely with systems subjected to these three forces, but instead of the linear-in-velocity damping forces being dissipative and described by positive definite damping matrices as in the KTC theorem, it considers linear-in-velocity damping forces that are indefinite and described by indefinite damping matrices.

In contrast to the KTC result, recent work on this subject has shown that gyroscopically stabilized unstable potential systems can be made stable and even exponentially stable, through the introduction of linear damping by way of indefinite damping matrices [5]. To prove this general principle, in Ref. [5] two-degree-of-freedom gyroscopically stabilized systems are considered. It is shown that when such (generic) unstable potential systems are gyroscopically stabilized, they can always be made not just stable but exponentially stable by the introduction of a suitable indefinite damping matrix. The practical implementation of such damping involves the somewhat nonintuitive simultaneous use of both positive and negative velocity feedback. These feedbacks constructively interact to bestow exponential stability on the gyroscopically stabilized system. A continuous connected region in the space of indefinite damping matrices is shown to exist and is explicitly delineated [5]. All indefinite damping matrices that lie within this region are shown to make the gyroscopically stabilized system exponentially stable, and the finite stability boundary is explicitly obtained. Although sufficient to prove the basic

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concept underlying the use of indefinite damping matrices to make gyroscopically stabilized systems exponentially stable, two-degree-of-freedom systems rarely arise in practical engineered systems and in nature, except when strong symmetries exist or when special constraints are enforced. This paper shows how (generic) multi-degree-of-freedom (MDOF) systems with unstable potential matrices can be gyroscopically stabilized and always made exponentially stable through the use of linear-in-velocity indefinite damping. The results are therefore applicable to real-life unstable potential systems. They provide a general methodology for such systems guaranteeing exponential stability.

There is little literature on the use of indefinite damping matrices in gyroscopically stabilized unstable potential systems where the aim is to make the gyroscopically stabilized system exponentially stable. Merkin's book [6] deals with an attempt to use an indefinite damping matrix for a gyroscopically stabilized monorail car. Merkin shows that by increasing the gyroscopic stabilization force, stability might be possible provided that certain conditions are satisfied. However, there has been a long-term interest from the aerospace and mechanical engineering communities in gyroscopic systems subjected to dissipative damping [6–10]. In Refs. [11–15] Kirillov takes a different approach and deals with gyroscopic stabilization of nonconservative systems. One of the central analytical themes in these references is the consideration of first approximations of perturbed eigenvalues around, what the author calls, exceptional points where the real part of the eigenvalues is zero. Using this perturbational approach, asymptotic stability regions using indefinite damping matrices are obtained as the circulatory contributions to the nonconservative forces become small and/or tend to zero. The present paper, however, does not deal with any perturbation approaches, or with nonconservative forces since, as mentioned before, such forces do not enter the KTC paradigm.

Indefinite damping matrices entail the use of positive velocity feedback, and such feedback has been shown to be useful in certain dynamic systems [16,17]; it is these papers that have inspired the present paper and Ref. [5].

We begin with the N-degree-of-freedom unstable potential system described by

$$\tilde{M}\,\ddot{q} + \tilde{K}q = 0 \tag{1}$$

where $q \in R^N$; $\tilde{M}$ is a real, constant, positive definite matrix mass matrix; and the potential matrix $\tilde{K}$ is a real, constant, symmetric stiffness matrix. We assume throughout this paper that $\tilde{K}$ has $2n$, $n \geq 1$, negative eigenvalues and $N - 2n \geq 0$ positive eigenvalues. The $2n$ negative eigenvalues of the potential matrix $\tilde{K}$ represent the unstable modes of vibration of the physical system modeled by Eq. (1).

Using the transformation $q(t) = \tilde{M}^{-1/2}y(t)$ in Eq. (1) and premultiplying both sides by $\tilde{M}^{-1/2}$, this equation becomes

$$\ddot{y} + \hat{K}y = 0 \tag{2}$$

where $\hat{K} = \tilde{M}^{-1/2}\tilde{K}\tilde{M}^{-1/2}$ is a symmetric matrix. Since $\hat{K}$ is symmetric there is a real orthogonal matrix T such that $T^T\hat{K}T = \Lambda$, where Λ is a diagonal matrix that has the eigenvalues of $\tilde{K}$ along its diagonal. Using the transformation $y(t) = Tx(t)$ in Eq. (2) and premultiplying both sides by T^T, we get the relation

$$\ddot{x} + \Lambda x = 0 \tag{3}$$

which describes the dynamics of the unstable N-degree-of-freedom potential system. Equation (3) is clearly equivalent to Eq. (1), and it is Eq. (3) that we will be using in this paper.

What is important for what follows is that the ith column, t_i, of the matrix T is an eigenvector of the matrix $\hat{K}$ that corresponds to the eigenvalue λ_i that sits in the ith row of the diagonal matrix Λ. Hence by rearranging the order in which the orthonormal columns of T are sequentially listed in the matrix T, the eigenvalues that lie along the diagonal of the matrix Λ in Eq. (3) can be placed in any desired order. We initially consider a row vector, K_p, containing the eigenvalues of the potential system given by

$$K_p = [K_{p-}, K_{p+}] = \Bigg[\underbrace{-\lambda_1, -\lambda_2, \ldots, -\lambda_{2n}}_{K_{p-}}, \underbrace{\lambda_{2n+1}, \ldots, \lambda_N}_{K_{p+}}\Bigg], \quad \text{with } \lambda_i > 0,\ \ i = 1, 2, \ldots, N \tag{4}$$

and label the eigenvalues so that $-\lambda_1 \leq -\lambda_2 \leq \cdots \leq -\lambda_{2n} < \lambda_{2n+1} \leq \cdots \leq \lambda_N$. The row vector $K_{p-} \coloneqq [-\lambda_1, -\lambda_2, \ldots, -\lambda_{2n}]$ contains the $2n$ negative eigenvalues of the potential matrix K, and the row vector $K_{p+} \coloneqq [\lambda_{2n+1}, \lambda_{2n+2}, \ldots, \lambda_N]$ contains the remaining positive eigenvalues of K. With no loss of generality, we then have

$$\Lambda = \mathrm{diag}(K_p) \tag{5}$$

in Eq. (3). We show later on how to order the negative eigenvalues contained in the row vector, K_{p-}; for now, we label the negative eigenvalues in ascending order. Note that we assume that zero is not an eigenvalue of $\hat{K}(\tilde{K})$.

Remark 1: That the eigenvalues along the diagonal of the matrix Λ of the (unstable) potential system can be placed in any desired order appears to be obvious. Yet, it will turn out to be of considerable importance later on when we relax the condition that the negative eigenvalues be placed in ascending order in K_{p-} (see Remark 5). Here we only note that with no loss of generality, the elements of K_{p-} can be placed in any order in Eqs. (3) and (5). □

We assume throughout this paper that the unstable potential system described by Eq. (3) [or equivalently the system described by Eq. (1)] is subjected to suitable gyroscopic forces described by a gyroscopic, skew-symmetric, matrix G so that it is stabilized. A simple way to ensure this for MDOF systems is provided later on in this paper. The motion of this undamped gyroscopically stabilized system is then given by

$$\ddot{x} + G\dot{x} + \Lambda x = 0 \tag{6}$$

The KCT paradigm informs us that the minutest dissipative damping added to the gyroscopically stabilized system described by Eq. (6) makes it unstable.

Remark 2: To make Eq. (6) more explicit, we illustrate it for an unstable six-degree-of-freedom potential system in which the 6-by-6 matrix $\hat{K}(\tilde{K})$ has four negative eigenvalues ($n = 2$). The equation, in expanded form, is

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \\ \ddot{x}_5 \\ \ddot{x}_6 \end{bmatrix} + \underbrace{\left[\begin{array}{c|c} \hat{G} & 0 \\ \hline 0 & 0 \end{array}\right]}_{G} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} + \underbrace{\left[\begin{array}{cccc|cc} -\lambda_1 & & & & & \\ & -\lambda_2 & & & & \\ & & -\lambda_3 & & 0 & \\ & & & -\lambda_4 & & \\ \hline & & 0 & & \lambda_5 & \\ & & & & & \lambda_6 \end{array}\right]}_{\Lambda} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = 0, \quad \lambda_i > 0,\ \ i = 1, 2, \ldots, 6 \tag{7}$$

where $K_{p-} = [-\lambda_1, -\lambda_2, -\lambda_3, -\lambda_4]$, $K_{p+} = [\lambda_5, \lambda_6]$, and $\hat{G}$ is a skew-symmetric matrix. Since $\lambda_5, \lambda_6 > 0$, the last two degrees of freedom, x_5 and x_6, are marginally stable and do not require any gyroscopic stabilization. We further express the matrix $\hat{G}$ as a block-diagonal matrix so that $\hat{G} = \mathrm{BLKdiag}(G_1, G_2)$, where G_1 and G_2 are each 2-by-2 skew-symmetric matrices. We use the notation $u_1(t) = [x_1(t), x_2(t)]^T$, $u_2(t) = [x_3(t), x_4(t)]^T$, $v(t) = [x_5(t), x_6(t)]^T$, $\Lambda_1 = \mathrm{diag}(-\lambda_1, -\lambda_2)$, $\Lambda_2 = \mathrm{diag}(-\lambda_3, -\lambda_4)$, and $\Lambda_s = \mathrm{diag}(\lambda_5, \lambda_6)$. The subscript s denotes the stable part of the potential system. Thus, $\Lambda = \mathrm{BLKdiag}(\Lambda_1, \Lambda_2, \Lambda_s)$, in which $\Lambda_i < 0$, $i = 1, 2$, and $\Lambda_s > 0$. Equation (7) thus comprises the three subsystems

$$\ddot{u}_i + G_i\dot{u}_i + \Lambda_i u_i = 0, \qquad \Lambda_i < 0, \quad i = 1, 2, \quad \text{and,}$$
$$\ddot{v} \quad + \Lambda_s v = 0, \qquad \Lambda_s > 0 \tag{8}$$

The coordinate $v(t)$ is marginally stable; the coordinates $u_1(t)$ and $u_2(t)$ can be made marginally stable through an appropriate choice of the matrices G_1 and G_2. □

Analogous to the six-degree-of-freedom system considered above, when the N-by-N matrix $\hat{K}(\tilde{K})$ has $2n \le N$ negative eigenvalues the N-by-N matrix G in Eq. (6) will again be block diagonal, made up of a $2n$-by-$2n$ skew-symmetric block, $\hat{G}$, followed by a lower $(N-2n)$-by-$(N-2n)$ zero diagonal block. For gyroscopic stabilization, we then consider $\hat{G} = \text{BLKdiag}(G_1, G_2, \ldots, G_n)$, in which G_i, $1 \le i \le n$, are each 2-by-2 skew-symmetric matrices. Similarly, the N-by-N diagonal matrix Λ is considered block diagonal; its upper block contains the $2n$ negative eigenvalues of $\hat{K}$ given in K_{p-}, and its lower block contains the remaining $(N-2n)$ positive eigenvalues. Thus, $\Lambda = \text{BLKdiag}(\Lambda_1, \Lambda_2, \ldots, \Lambda_n, \Lambda_s)$, in which $\Lambda_i < 0$, $1 \le i \le n$, are 2-by-2 diagonal matrices each of which contains the negative eigenvalues of $\hat{K}$, and $\Lambda_s > 0$ is an $(N-2n)$-by-$(N-2n)$ diagonal matrix containing its positive eigenvalues. The last $(N-2n)$ degrees of freedom of the system are all marginally stable.

The aim of the paper can now be precisely stated. It is known that for a given generic unstable N-degree-of-freedom potential system described by Eq. (3) that has $2n \le N$ negative eigenvalues, there are an uncountably infinite number of (skew-symmetric) matrices G that gyroscopically stabilize the system. Having selected a matrix $\bar{G} \coloneqq G$ that gyro-stabilizes the system, our goal is to show that there are an uncountably infinite number of indefinite damping matrices D that guarantee that the damped gyroscopically stabilized system

$$\ddot{x} + (D + \bar{G})\dot{x} + \Lambda x = 0 \tag{9}$$

is *exponentially stable*. The dynamic system in Eq. (9) is thus made exponentially stable by choosing any one of the indefinite matrices $\bar{D} \coloneqq D$ that exponentially stabilizes it. What exactly is meant by the word "generic" in this context will be made clear later on.

In what follows, the system in Eq. (6) in which $G = \bar{G}$ is called the undamped gyroscopically stabilized system, and the one in Eq. (9) is called the damped gyroscopically stabilized system.

Remark 3: We note that $x(t) = (T^T \tilde{M}^{1/2})q(t) \coloneqq Wq(t)$. Once a skew-symmetric $\bar{G} \coloneqq G$ is chosen to gyro-stabilize the unstable potential system [see Eq. (6)], and an indefinite matrix $\bar{D} \coloneqq D$ is chosen to make this gyroscopically stabilized system exponentially stable [see Eq. (9)], the system described by

$$\tilde{M}\ddot{q} + (\tilde{D} + \tilde{G})\dot{q} + \tilde{K}q = 0 \tag{10}$$

is made exponentially stable. In Eq. (10) the indefinite damping matrix $\tilde{D}$ in terms of the coordinate q is $\tilde{D} = W^T\bar{D}W$, the gyroscopic matrix $\tilde{G} = W^T\bar{G}W$, and $\tilde{K} = W^T\Lambda W$.

Thus, once $\bar{G}$ and then $D = \bar{D}$ are found so that Eq. (9) is exponentially stable, $\tilde{D}$ and $\tilde{G}$ can be found for the unstable potential system in Eq. (1). By Sylvester's law of inertia, $\tilde{K}$ has the same signature as Λ, the matrix $\tilde{D}$ has the same signature as $\bar{D}$, and likewise the matrix $\tilde{G}$. □

II. Main Results

As a prelude to the results to be obtained in this paper, we briefly provide, in the lemma below, results hereto known when the unstable potential system has just two degrees of freedom (for a proof, see Ref. [5]). Thus, $n = 1$, $N = 2$, and the 2-by-2 diagonal matrix Λ has two negative eigenvalues $-\lambda_1$, and $-\lambda_2$; the row vector $K_p = K_{p-} = [-\lambda_1, -\lambda_2]$.

Lemma 1: Consider a two-degree-of-freedom unstable potential system

$$\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix} + \underbrace{\begin{bmatrix}-\lambda_1 & 0\\ 0 & -\lambda_2\end{bmatrix}}_{\Lambda}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = 0 \tag{11}$$

in which the parameters $\lambda_1, \lambda_2 > 0$ are known. We introduce the gyroscopic matrix G so the system

$$\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix} + \underbrace{\begin{bmatrix}0 & g\\ -g & 0\end{bmatrix}}_{G}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} + \underbrace{\begin{bmatrix}-\lambda_1 & 0\\ 0 & -\lambda_2\end{bmatrix}}_{\Lambda}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = 0 \tag{12}$$

is made stable. This requires that [5]

1) $\lambda_1, \lambda_2 > 0$, and

2) $g^2 = \lambda_1 + \lambda_2 + 2\sqrt{\lambda_1\lambda_2} + \delta^2, \quad \text{for any } \delta \ne 0$ (13)

The first of these conditions is met. The second condition can be rewritten as [see Eq. (11)]

$$g^2 = -\text{Trace}(\Lambda) + 2\sqrt{\text{Det}(\Lambda)} + \delta^2, \quad \text{for any } \delta \ne 0 \tag{14}$$

$\text{Det}(X)$ denotes the determinant of X. As seen, the value of g needed for gyroscopic stabilization depends on the values of $-\lambda_1$ and $-\lambda_2$, and on the choice of δ. By choosing a suitable value of $\bar{\delta} \coloneqq \delta \ne 0$, a value of $\bar{g} \coloneqq g$ that satisfies Eq. (13) can be obtained, and the system described in Eq. (12) therefore gyroscopically stabilized; however, only marginal stability can be achieved; exponential stability is not possible [5]. The chosen gyroscopically stabilized system (with $\bar{g} \coloneqq g$ and $\bar{\delta} \coloneqq \delta \ne 0$) in Eq. (12) is thus succinctly described by the ordered quadruple $\bar{r} \coloneqq \{\lambda_1, \lambda_2, \bar{g}, \bar{\delta}\}$. Given the quadruple $\bar{r}$, our aim now is to find an indefinite damping matrix, D^I, so that the damped gyroscopically stabilized system is exponentially stable (superscript I for "indefinite" damping).

We consider the following three cases, which depend on the values of λ_1 and λ_2 of the unstable potential matrix Λ.

Case 1: When the gyroscopically stabilized system $\bar{r} \coloneqq \{\lambda_1, \lambda_2, \bar{g}, \bar{\delta}\}$ has $-\lambda_1 < -\lambda_2 < 0$

The damped system [5]

$$\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix} + \underbrace{\begin{bmatrix}d & 0\\ 0 & -\alpha\end{bmatrix}}_{D^I}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} + \underbrace{\begin{bmatrix}0 & \bar{g}\\ -\bar{g} & 0\end{bmatrix}}_{\bar{G}}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} + \underbrace{\begin{bmatrix}-\lambda_1 & 0\\ 0 & -\lambda_2\end{bmatrix}}_{\Lambda}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = 0, \quad d, \alpha > 0 \tag{15}$$

is to be made exponentially stable by adding an appropriate indefinite damping matrix D^I to the system $\bar{r}$. Reference [5] shows that there are an uncountably infinite number of indefinite damping matrices D^I that will render this system exponentially stable. The system in Eq. (15) can be described by the ordered sextuple $s = \{\lambda_1, \lambda_2, \bar{g}, \bar{\delta}, -\alpha, d\}$. For a more detailed description of this notation see Appendix A. The values of the first four parameters of s are the same as those in the quadruple $\bar{r}$ that describes the specific undamped gyroscopically stabilized system; therefore, the sextuple s too has $-\lambda_1 < -\lambda_2 < 0$; the first element, λ_1, of s is greater than its second element, λ_2. The (1,1) and (2,2) elements of D^I correspond to the sixth and fifth elements, respectively, of the sextuple s.

The way to explicitly find the region of exponential stability in the $\alpha - d$ plane of a system $s = \{\lambda_1, \lambda_2, \bar{g}, \bar{\delta}, -\alpha, d\}$ whose first element, λ_1, is *greater* than its second element, λ_2, is summarized in Appendix A. This region of stability is continuous and connected [5].

By choosing any point with coordinates $\bar{\alpha} \coloneqq \alpha$ and $\bar{d} \coloneqq d$ that lies within this region of exponential stability, the system described by $\bar{s} = \{\lambda_1, \lambda_2, \bar{g}, \bar{\delta}, -\bar{\alpha}, \bar{d}\}$ in which $-\lambda_1 < -\lambda_2 < 0$, or alternatively, the system described in extensio by the equation [see Eq. (15)]

$$\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix}+\underbrace{\begin{bmatrix}\bar{d} & 0\\ 0 & -\alpha\end{bmatrix}}_{D^I}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}+\underbrace{\begin{bmatrix}0 & \bar{g}\\ -\bar{g} & 0\end{bmatrix}}_{\bar{G}}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}+\underbrace{\begin{bmatrix}-\lambda_1 & 0\\ 0 & -\lambda_2\end{bmatrix}}_{\Lambda}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=0,$$

$$-\lambda_1<-\lambda_2<0,\ \bar{d},\bar{\alpha}>0 \tag{16}$$

is guaranteed to be exponentially stable.

Case 2: When the gyroscopically stabilized system $\bar{r} \coloneqq \{\lambda_1, \lambda_2, \bar{g}, \bar{\delta}\}$ has $-\lambda_2 < -\lambda_1 < 0$

The damped system [5]

$$\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix}+\underbrace{\begin{bmatrix}-\alpha & 0\\ 0 & d\end{bmatrix}}_{D^I}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}+\underbrace{\begin{bmatrix}0 & \bar{g}\\ -\bar{g} & 0\end{bmatrix}}_{\bar{G}}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}+\underbrace{\begin{bmatrix}-\lambda_1 & 0\\ 0 & -\lambda_2\end{bmatrix}}_{\Lambda}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=0,\quad d,\alpha>0 \tag{17}$$

is made exponentially stable by adding an indefinite damping matrix D^I to the system $\bar{r}$ with a proper choice of the parameters d and $-\alpha$. A continuous, connected region in the $\alpha - d$ plane exists for which the system is exponentially stable [5]. The system in Eq. (17) can be written as $s' = \{\lambda_1, \lambda_2, \bar{g}, \bar{\delta}, d, -\alpha\}$, in which the first four element of s' are the same as those in $\bar{r}$; hence, the first element, λ_1, of s' is now less than its second, λ_2. Note that now the (1,1) element of D^I in Eq. (17) is $-\alpha$, and its (2,2) element is d. This is reflected in the sixth and fifth element of s', respectively. Compare the matrices D^I in Eqs. (15) and (17).

In Ref. [5] it is shown that the region of exponential stability in the $\alpha - d$ plane of the system $s' = \{\lambda_1, \lambda_2, \bar{g}, \bar{\delta}, d, -\alpha\}$ in which $-\lambda_2 < -\lambda_1 < 0$ is the same as that of the system $s^* = \{\lambda_2, \lambda_1, \bar{g}, \bar{\delta}, -\alpha, d\}$. Appendix A shows how to find the region of stability in the $\alpha - d$ plane of system s^* whose first element, λ_2, is now greater than its second, λ_1. This region of stability is explicitly determined as shown in Appendix A in the same manner as in Case 1.

Thus, using Appendix A we first find the region of exponential stability for the system s^* in the $\alpha - d$ plane, and then choose any point with coordinates $\bar{\alpha} \coloneqq \alpha$ and $\bar{d} \coloneqq d$ within this region. Having chosen this point, the system $\bar{s}' = \{\lambda_1, \lambda_2, \bar{g}, \bar{\delta}, \bar{d}, -\bar{\alpha}\}$ [Eq. (17)]

$$\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix}+\underbrace{\begin{bmatrix}-\bar{\alpha} & 0\\ 0 & \bar{d}\end{bmatrix}}_{D^I}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}+\underbrace{\begin{bmatrix}0 & \bar{g}\\ -\bar{g} & 0\end{bmatrix}}_{\bar{G}}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}+\underbrace{\begin{bmatrix}-\lambda_1 & 0\\ 0 & -\lambda_2\end{bmatrix}}_{\Lambda}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=0,$$

$$-\lambda_2<-\lambda_1<0,\ \bar{d},\ \bar{\alpha}>0 \tag{18}$$

is then guaranteed to be exponentially stable.

Case 3: When the gyroscopically stabilized system is described by $\bar{r} \coloneqq \{\lambda_1, \lambda_2, \bar{g}, \bar{\delta}\}$ with $-\lambda_1 = -\lambda_2 < 0$

In this case, though stability can still be achieved by the addition of an indefinite matrix D^I with $\alpha = \delta$, only marginal stability is possible, and not asymptotic stability. This situation is nongeneric. In complex engineered and naturally occurring physical systems that are often modeled by hundreds, and commonly by thousands, of degrees of freedom, such multiple (or close) eigenvalues are more likely to arise. However, as shown in Sec. II.B, for MDOF systems in which some of the negative eigenvalues may be equal, this difficulty can almost always be circumvented, thus allowing such systems also to be made exponentially stable through the introduction of indefinite damping (matrices). □

Remark 4: Equations (15) and (17) provide the following observation. Dissipative damping $d > 0$ in D^I, or negative velocity feedback, is provided to the degree of freedom that has the lower of the values among $-\lambda_1$ and $-\lambda_2$ in the potential matrix. The degree of freedom that has the higher value among $-\lambda_1$ and $-\lambda_2$ receives positive velocity feedback, namely, $-\alpha$ ($\alpha > 0$) in D^I.

The positive and negative velocity feedbacks that are required to produce this indefinite damping can be thought of as the control needed to achieve exponential stability of the gyroscopically stabilized system. Thus, to achieve exponential stability of the gyroscopically stabilized system, one requires the simultaneous use of both positive and negative velocity feedback. It is through the interaction of these competing feedbacks that the system acquires exponential stability. □

A. Distinct Negative Eigenvalues of the Potential Matrix $\hat{K}$

With this lemma in hand, we proceed to the N-degree-of-freedom system in Eq. (3) in which the negative eigenvalues of the unstable potential matrix are distinct. In the subsection that follows, we permit the negative eigenvalues of Λ to be nondistinct.

Result 1: Consider a given unstable N-degree-of-freedom potential system

$$\ddot{x} + \Lambda x = 0 \tag{19}$$

in which Λ is a diagonal matrix having $2n \le N$ distinct negative eigenvalues, and $N - 2n > 0$ positive eigenvalues.

The gyroscopically stabilized system described by

$$\ddot{x} + G\dot{x} + \Lambda x = 0 \tag{20}$$

can be made marginally stable by an uncountably infinite number of skew-symmetric matrices G. Having picked a particular skew-symmetric matrix $\bar{G} \coloneqq G$ that bestows gyroscopic stability, the gyroscopically stabilized system

$$\ddot{x} + \bar{G}\,\dot{x} + \Lambda x = 0 \tag{21}$$

can always be made exponentially stable using a suitable indefinite damping matrix D. A methodology is developed to find an uncountable set of damping matrices D so that the damped gyroscopically stabilized system

$$\ddot{x} + D\dot{x} + \bar{G}\,\dot{x} + \Lambda x = 0 \tag{22}$$

can always be made exponentially stable for the given matrix $\bar{G}$ that makes the system in Eq. (21) gyroscopically stable.

Proof: The $2n$ negative diagonal elements of the potential matrix Λ being distinct, with no loss of generality they can be ordered so that

$$-\lambda_1 < -\lambda_2 < \cdots < -\lambda_{2n} < \lambda_{2n+1} \le \cdots \le \lambda_N \tag{23}$$

where $\lambda_i > 0$, $i = 1, 2, \ldots, N$. Later on, this ordering will be relaxed.

We can now take

$$K_p = [K_{p-}, K_{p+}] = \Big[\underbrace{-\lambda_1, -\lambda_2, \ldots, -\lambda_{2n}}_{K_{p-}}, \underbrace{\lambda_{2n+1}, \ldots, \lambda_N}_{K_{p+}}\Big] \tag{24}$$

so that $\Lambda = \mathrm{diag}(K_p)$.

The methodology for obtaining exponential stability of the gyroscopically stabilized unstable potential system described in Eq. (21) is carried out in two conceptual steps: 1) starting with Eq. (19), we find the set of gyroscopic matrices G that ensure that Eq. (20) is marginally stable, and 2) having picked a suitable matrix $\bar{G} = G$ that belongs to this set, we find a suitable matrix D that makes Eq. (22) exponentially stable.

The unstable potential matrix Λ is split into two parts: one that has all the negative eigenvalues, and the other that has all the positive eigenvalues. The unstable part, which has $2n$ degrees of freedom, is further expressed as a block diagonal matrix of n different two-degree-of-freedom uncoupled subsystems. These n subsystems are gyroscopically stabilized and made marginally stable. They are then made exponentially stable by using an indefinite damping matrix (Lemma 1, Case 1). The stable part of the potential matrix, Λ_s, has $N - 2n$ degrees of freedom and is marginally stable. It can be made exponentially stable by using any positive-definite dissipative damping matrix.

We denote the 2-by-2 block diagonal matrices

$$\Lambda_i = \begin{bmatrix} -\lambda_{2i-1} & 0 \\ 0 & -\lambda_{2i} \end{bmatrix} := \mathrm{diag}(-\lambda_{2i-1}, -\lambda_{2i}), \qquad \text{and}$$

$$G_i = \begin{bmatrix} 0 & g_i \\ -g_i & 0 \end{bmatrix} := \mathrm{Skew}(g_i),\ i = 1, 2, \ldots, n \tag{25}$$

and the diagonal matrix

$$\Lambda_s = \mathrm{diag}(K_{p+}) = \mathrm{diag}(\lambda_{2n+1}, \ldots \lambda_N) \tag{26}$$

Note that, by assumption, each of the matrices Λ_i in Eq. (25) has distinct negative eigenvalues.

The N-by-N diagonal matrix Λ and the N-by-N skew-symmetric matrix G in Eq. (20) can then be written as the block diagonal matrices,

$$\Lambda = \mathrm{BLKdiag}(\Lambda_u, \Lambda_s) = \mathrm{BLKdiag}(\Lambda_1, \Lambda_2, \ldots \Lambda_n, \Lambda_s), \quad \text{and}$$

$$G = \mathrm{BLKdiag}(G_1, G_2, \ldots G_n, G_{n+1}), \qquad G_{n+1} = 0 \tag{27}$$

where *BLKdiag* denotes block diagonal. The n submatrices G_i, $i = 1, 2, \ldots, n$, are each 2-by-2; the last block, G_{n+1}, of G is an $(N-2n)$-by-$(N-2n)$ zero matrix. The subscripts u and s on Λ denote the unstable and stable parts of the potential matrix, respectively.

Denoting $u_i = [x_{2i-1}, x_{2i}]^T$, $i = 1, 2, \ldots, n$, and $v = [x_{2n+1}, \ldots x_N]^T$, Eq. (20) can now be written compactly as a set of the $N - n$ uncoupled subsystems:

$$\ddot{u}_i + G_i \dot{u}_i + \Lambda_i u_i = 0,\ i = 1, 2, \ldots n \tag{28}$$

$$\ddot{v} + \Lambda_s v = 0 \tag{29}$$

Equation (28) consists of n two-degree-of-freedom gyroscopic potential subsystems that are decoupled from one another; the ith gyroscopic subsystem is described by the three parameters $-\lambda_{2i-1}, -\lambda_{2i}$, and g_i given in Eq. (25). Equation (29) consists of $N - 2n$ uncoupled single-degree-of-freedom marginally stable systems.

We know (Lemma 1) that by choosing

$$g_i = \sqrt{-\mathrm{Trace}(\Lambda_i) + 2\sqrt{Det(\Lambda_i)} + \delta_i^2},\ \text{with } \delta_i \neq 0,\ i = 1, 2, \ldots, n \tag{30}$$

each of the n subsystems in Eq. (28) can be made marginally stable. Each choice of $\delta_i \neq 0$, $1 \leq i \leq n$, corresponds to a value of g_i, $1 \leq i \leq n$, obtained from Eq. (30). The g_i, $1 \leq i \leq n$, that satisfy Eq. (30) are therefore uncountably infinite. Each g_i so chosen gives a corresponding 2-by-2 skew-symmetric matrix $G_i = \mathrm{Skew}(g_i)$, $1 \leq i \leq n$. Hence, there are an uncountably infinite number of G_i's and therefore an uncountably infinite number of matrices G [see Eq. (27)] that make the potential system marginally stable.

By choosing a specific set of $\bar{\delta}_i(\bar{g}_i) := \delta_i(g_i)$, $i = 1, 2, \ldots, n$ so that they satisfy Eq. (30), the ith gyroscopic subsystem, $1 \leq i \leq n$, is made marginally stable. We denote the parameters chosen that describe the stabilized ith subsystem, $1 \leq i \leq n$, in Eq. (28) by the ordered quadruple $\bar{r}_i := \{\lambda_{2i-1}, \lambda_{2i}, \bar{g}_i, \bar{\delta}_i\}$. With the quadruples so chosen, the gyroscopically stabilized system [Eq. (21)] can then be written as

$$\ddot{u}_i + \bar{G}_i \dot{u}_i + \Lambda_i u_i = 0, \quad i = 1, 2, \ldots n \tag{31}$$

$$\ddot{v} + \qquad \Lambda_s v = 0 \tag{32}$$

Here $\bar{G} = \mathrm{BLKdiag}(\bar{G}_1, \bar{G}_2, \ldots \bar{G}_n, G_{n+1})$, $G_{n+1} = 0$,

$$\bar{G}_i = \mathrm{Skew}(\bar{g}_i),\ i = 1, 2, \ldots n \tag{33}$$

Since $\Lambda_s > 0$, each of the $N - 2n$ equations in Eq. (32) is also marginally stable, and therefore there is no need to apply any gyroscopic force to this subsystem; that is why the lowest block, G_{n+1}, of the matrix G may be taken to be zero.

Noting that each subsystem $\bar{r}_i := \{\lambda_{2i-1}, \lambda_{2i}, \bar{g}_i, \bar{\delta}_i\}$, $1 \leq i \leq n$, has $-\lambda_{2i-1} < -\lambda_{2i} < 0$, we now introduce the indefinite damping matrices

$$D_i^I = \begin{bmatrix} d_i & 0 \\ 0 & -\alpha_i \end{bmatrix} := \mathrm{diag}(d_i, -\alpha_i),\ d_i, \alpha_i > 0,\ i = 1, 2, \ldots, n \tag{34}$$

along with the matrix

$$D_s^{\mathrm{Diss}} = \mathrm{diag}(c_1, c_2, \ldots c_{N-2n}),\ \ c_i > 0,\ \ i = 1, 2, \ldots, N - 2n \tag{35}$$

where the superscript I denotes indefinite damping matrices in Eq. (34), and the superscript *Diss* denotes the $(N-2n)$-by-$(N-2n)$ dissipative damping matrix in Eq. (35). The constants c_i can be suitably chosen to apportion a desired percentage of critical damping to each of the uncoupled $N - 2n$ marginally stable modes of vibration of the potential subsystem shown in Eq. (32) in order to make them asymptotically stable.

The resulting damped gyroscopically stabilized system is

$$\ddot{u}_i + D_i^I \dot{u}_i + \bar{G}_i \dot{u}_i + \Lambda_i u_i = 0,\ \ i = 1, 2, \ldots n \tag{36}$$

$$\ddot{v} + D_s^{\mathrm{Diss}} \dot{v} + \qquad \Lambda_s v = 0 \tag{37}$$

In Eq. (36) there are n uncoupled damped gyroscopically stabilized two-degree-of-freedom subsystems that now need to be made exponentially stable. The ith, $1 \leq i \leq n$, such marginally stable subsystem is described by the quadruple $\bar{r}_i := \{\lambda_{2i-1}, \lambda_{2i}, \bar{g}_i, \bar{\delta}_i\}$, $1 \leq i \leq n$, whose elements are all known. To ensure its exponential stability a proper choice of the parameters (α_i, d_i) is required to be made to specify the matrix D_i^I [Eq. (34)]. Since the eigenvalues of the matrix Λ_i are distinct with $-\lambda_{2i-1} < -\lambda_{2i} < 0$, Lemma 1 guarantees that exponential stability of each of the n subsystems in Eq. (36) is always possible.

Recall that $u_i = [x_{2i-1}, x_{2i}]^T$, $i = 1, 2, \ldots, n$. Hence, by Case 1 in Lemma 1 [see Eqs. (11) and (25)], in each of the n subsystems in Eq. (36), the degrees of freedom x_{2i-1}, $i = 1, 2, \ldots, n$, receive dissipative damping (or negative velocity feedback), whereas the degrees of freedom x_{2i}, $i = 1, 2, \ldots, n$, simultaneously receive positive feedback, as shown by the structure of D_i^I given in Eq. (34).

Knowing the parameters in the quadruple $\bar{r}_i$ that describes the specific ith gyroscopically stabilized system, the region of exponential stability in the $\alpha - d$ plane for this ith subsystem, $1 \leq i \leq n$, in Eq. (36) can be explicitly found (see Appendix A). Any point $\bar{d}_i := d_i$ and $\bar{\alpha}_i := \alpha_i$ can now be chosen within this region of exponential stability so that the ith subsystem, $1 \leq i \leq n$, is exponentially stable. This ith damped exponentially stable system is then described by the ordered sextuple $\bar{s}_i = \{\lambda_{2i-1}, \lambda_{2i}, \bar{g}_i, \bar{\delta}_i, -\bar{\alpha}_i, \bar{d}_i\}$. We denote $\bar{D}_i^I := \mathrm{diag}(\bar{d}_i, -\bar{\alpha}_i)$, $i = 1, 2, \ldots, n$.

That the $N - 2n$ uncoupled single-degree-of-freedom subsystems shown in Eq. (37) are each exponentially stable is trivial to show. This is because the diagonal matrices D_s^{Diss} and Λ_s are positive definite, the former being so because it is diagonal, with elements $c_i > 0$, $i = 1, 2, \ldots, N - 2n$. Such a purely dissipative matrix can also be brought about by simply using negative velocity feedback.

With the damping matrices so determined, the N-degree-of-freedom unstable potential system, which is gyroscopically stabilized as in Eqs. (31) and (32), is rendered exponentially stable. That is, the system

$$\ddot{u}_i + \bar{D}_i^I \dot{u}_i + \bar{G}_i \dot{u}_i + \Lambda_i u_i = 0,\ i = 1, 2, \ldots n \tag{38}$$

$$\ddot{v} + D_s^{\mathrm{Diss}} \dot{v} + \qquad \Lambda_s v = 0 \tag{39}$$

is guaranteed to have exponentially stability, and hence the system in Eq. (10) is guaranteed to be exponentially stable. □

Remark 5: As seen from the proof, the main issue is bringing about exponential stability in the n gyroscopically stabilized, marginally stable two-degree-of-freedom subsystems shown in Eq. (31), which contain the $2n$ negative eigenvalues of the unstable potential matrix Λ. And for doing this, as we shall see later (Result 3), we really do not need all the negative eigenvalues of Λ to be distinct from one another; all that is required by Lemma 1 is that the negative eigenvalues (or diagonal elements) contained in each of the 2-by-2 matrices Λ_i, $i = 1, 2, \ldots, n$ in Eq. (31) be distinct. □

Remark 6: Instead of using a diagonal matrix D_s^{Diss} [see Eq. (35)] in Eq. (39) one could use any $N - 2n$ matrix $D_s^{\text{Diss}} > 0$; this would, in general, couple the components of the vector v. □

Numerical Example 1: Consider the 11-degree-of-freedom unstable potential system

$$\ddot{x} + \Lambda x = 0 \tag{40}$$

where $\Lambda = \text{diag}(-6, -5.25, -4.5, -3.75, -3, -2.25, -1.75, -1.25, 1, 3, 5)$. Thus, $N = 11$, and $n = 4$.

The row vectors

$$K_{p-} = [\overbrace{-6, -5.25}^{(-\lambda_1, -\lambda_2)}, \overbrace{-4.5, -3.75}^{(-\lambda_3, -\lambda_4)}, \overbrace{-3, -2.25}^{(-\lambda_5, -\lambda_6)}, \overbrace{-1.75, -1.25}^{(-\lambda_7, -\lambda_8)}] \text{ and}$$
$$K_{p+} = [\overset{\lambda_9}{1} \;\; \overset{\lambda_{10}}{3} \;\; \overset{\lambda_{11}}{5}] \tag{41}$$

list the negative and the positive eigenvalues in ascending order. Note that elements of the row vector K_{p-} are all distinct. In Sec. II.B, we will consider systems when this is not so.

The eigenvalues in K_{p-} are paired by the square brackets in Eq. (41) so that we have 4 subsystems, each with two degrees of freedom. The unstable potential matrix is $\Lambda = \text{BLKdiag}\,(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_s)$ in which

$$\Lambda_1 = \text{diag}(-6.5, -5.25), \quad \Lambda_2 = \text{diag}(-4.5, -3.75),$$
$$\Lambda_3 = \text{diag}(-3, -2.25), \quad \Lambda_4 = (-1.75, -1.25), \quad \Lambda_s = \text{diag}(1, 3, 5) \tag{42}$$

To gyroscopically stabilize each of these two-degree-of-freedom, potential subsystems we introduce five gyroscopic (skew-symmetric) matrices:

$$G_i = \text{Skew}(g_i), \; i = 1, 2, 3, 4, \; \text{and } G_5 = \text{diag}(0, 0, 0) \tag{43}$$

The 11-degree-of-freedom gyroscopic potential system $\ddot{x} + G\dot{x} + \Lambda x = 0$, which is described by Eqs. (28) and (29), can now be written in the expanded block-diagonal form as

$$\underbrace{\begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \\ \ddot{u}_4 \\ \ddot{v} \end{bmatrix}}_{\ddot{x}} + \underbrace{\begin{bmatrix} G_1 & & & & \\ & G_2 & & & \\ & & G_3 & & \\ & & & G_4 & \\ & & & & 0 \end{bmatrix}}_{G} \underbrace{\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \\ \dot{u}_4 \\ \dot{v} \end{bmatrix}}_{\dot{x}} + \underbrace{\begin{bmatrix} \Lambda_1 & & & & \\ & \Lambda_2 & & & \\ & & \Lambda_3 & & \\ & & & \Lambda_4 & \\ & & & & \Lambda_s \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v \end{bmatrix}}_{x} = 0 \tag{44}$$

where we denote the two vectors $u_1(t) \coloneqq [x_1, x_2]^T, u_2(t) \coloneqq [x_3, x_4]^T, u_3(t) \coloneqq [x_5, x_6]^T, u_4(t) \coloneqq [x_7, x_8]^T$, and the 3-vector $v(t) \coloneqq [x_9, x_{10}, x_{11}]^T$. As before, the uncoupled three-degree-of-freedom system described by the 3-vector v is already marginally stable, and needs no gyroscopic intervention; hence, the lowest block, G_5, along the diagonal of the matrix G is zero.

The other four uncoupled gyroscopic subsystems in Eq. (44) can be stabilized (see Lemma 1) by choosing the elements g_i in each of the (nonzero) matrices G_i so that

$$g_i = \sqrt{-\text{Trace}(\Lambda_i) + 2\sqrt{\text{Det}(\Lambda_i)} + \delta_i^2}, \text{ with } \delta_i \neq 0, \; i = 1, 2, \ldots, n \tag{45}$$

Using the parameters given in the matrices Λ_i, $i = 1, 2, 3, 4$, we take the following values $\bar{g}_i \coloneqq g_i$, which satisfy Eq. (45):

$$\bar{g}_1 = \sqrt{35} \approx 5.92, \quad \bar{g}_2 = \sqrt{25} = 5, \quad \bar{g}_3 = \sqrt{15} \approx 3.87, \quad \text{and}$$
$$\bar{g}_4 = \sqrt{10} \approx 3.16 \tag{46}$$

With these values of $\bar{g}_i$, $i = 1, 2, 3, 4$, the system shown in Eq. (44) with $G_i = \bar{G}_i = \text{Skew}(\bar{g}_i)$, $i = 1, 2, 3, 4$, is now marginally stable. The quadruples that describe these gyroscopically stabilized subsystems are

$$\bar{r}_1 = \{6, 5.25, \sqrt{35}, 3.53\}, \quad \bar{r}_2 = \{4.5, 3.75, 5, 2.92\},$$
$$\bar{r}_3 = \{3, 2.25, \sqrt{15}, 2.13\}, \quad \text{and} \quad \bar{r}_4 = \{1.75, 1.25, \sqrt{10}, 2.01\} \tag{47}$$

We now determine the indefinite diagonal matrices $D_i^I = \text{diag}(d_i, -\alpha_i)$, $i = 1, 2, 3, 4$, and the matrix D_s^{Diss}, so that this linear undamped gyroscopically stabilized system is guaranteed to be exponentially stable.

The expanded block diagonal form of the damped gyroscopically stabilized system is [see Eqs. (36) and (37)]

$$\begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \\ \ddot{u}_4 \\ \ddot{v} \end{bmatrix} + \underbrace{\begin{bmatrix} D_1^I + \bar{G}_1 & & & & \\ & D_2^I + \bar{G}_2 & & & \\ & & D_3^I + \bar{G}_3 & & \\ & & & D_4^I + \bar{G}_4 & \\ & & & & D_s^{\text{Diss}} \end{bmatrix}}_{D + \bar{G}} \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \\ \dot{u}_4 \\ \dot{v} \end{bmatrix}$$
$$+ \underbrace{\begin{bmatrix} \Lambda_1 & & & & \\ & \Lambda_2 & & & \\ & & \Lambda_3 & & \\ & & & \Lambda_4 & \\ & & & & \Lambda_s \end{bmatrix}}_{\Lambda} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v \end{bmatrix} = 0 \tag{48}$$

Making the decoupled response $v(t)$ of the lowest subsystem in Eq. (48) exponentially stable is trivial; one can use any values to $c_i > 0$, $i = 1, 2, 3$ with $D_s^{\text{Diss}} = \text{diag}(c_1, c_2, c_3)$. For example, these parameters can be chosen to be $c_1 = 0.01$, $c_2 = 0.03$, $c_3 = 0.06$, so that the dissipative damping matrix to damp out the stable modes of vibration of the potential system is then

$$D_s^{\text{Diss}} = \text{diag}(0.01, 0.03, 0.06) \tag{49}$$

To make the response $u_i(t)$, $i = 1, 2, 3, 4$, of the other four gyro-stabilized two-degree-of-freedom subsystems exponentially stable, the stability regions in the $\alpha - d$ plane for each of these four decoupled gyroscopically stabilized subsystems that are described by the quadruples $\bar{r}_i$, $i = 1, 2, 3, 4$, must be obtained. These fan-shaped stability regions for each of these four gyroscopically stabilized subsystems $\bar{r}_i$, $i = 1, 2, 3, 4$, given in Eq. (47) are shown in Fig. 1. The explicit determination of these regions of exponential

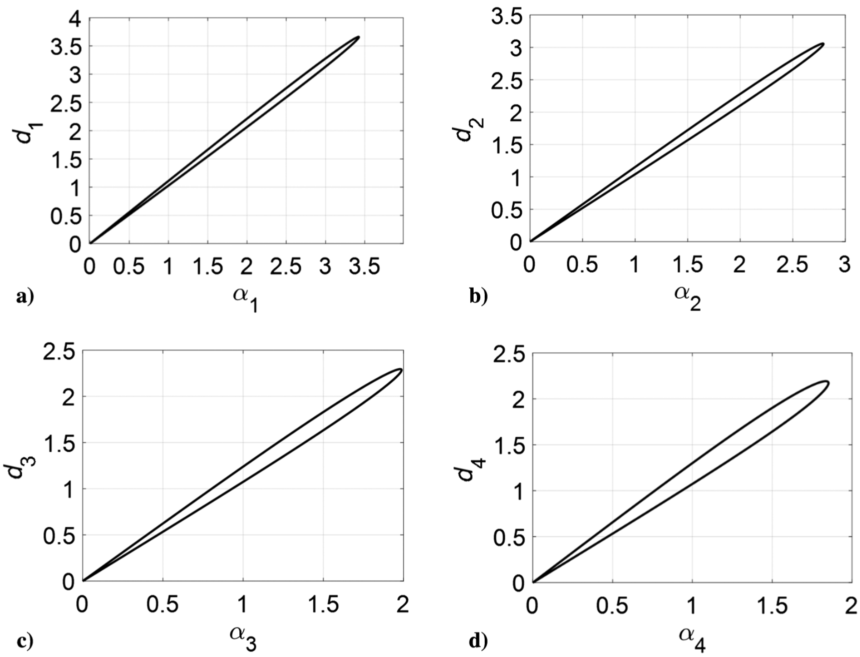

Fig. 1 Regions of exponential stability for the four gyroscopically stabilized systems: a) $\bar{r}_1(t)$, b) $\bar{r}_2(t)$, c) $\bar{r}_3(t)$, and d) $\bar{r}_4(t)$.

stability is explained in the Appendix A (in which Case 1 is applicable here).

The choice of any point $\bar{\alpha}_i \coloneqq \alpha_i, \bar{d}_i \coloneqq d_i\ i = 1, 2, 3, 4$, that lies in the interior of the fan-shaped stability loop for the system described by the quadruple $\bar{r}_i$, $i = 1, 2, 3, 4$, when used in the damping matrix $D_i^I = \mathrm{diag}(\bar{d}_i, -\bar{\alpha}_i)$, $i = 1, 2, 3, 4$, assures exponential stability of the corresponding response $u_i(t)$ of the damped gyroscopically stabilized subsystem [see Eqs. (36) and (48)]

$$\ddot{u}_i + (D_i^I + \bar{G}_i)\dot{u}_i + \Lambda_i u_i = 0,\ i = 1, 2, 3, 4 \tag{50}$$

For illustration, four points are taken that lie inside each of the four stability regions shown in Fig. 1 for each of the four uncoupled gyroscopically stabilized systems $\bar{r}_i, i = 1, 2, 3, 4$, in Eq. (47). They are

$$(\bar{\alpha}_1, \bar{d}_1) = (0.4, 0.43),\ (\bar{\alpha}_2, \bar{d}_2) = (0.15, 0.165), (\bar{\alpha}_3, \bar{d}_3) = (0.1, 0.115),\ \text{and}\ (\bar{\alpha}_4, \bar{d}_4) = (0.15, 0.18) \tag{51}$$

so that the matrices

$$\bar{D}_1^I = \mathrm{diag}(0.43, -0.4), \qquad \bar{D}_2^I = \mathrm{diag}(0.165, -0.15),$$
$$\bar{D}_3^I = \mathrm{diag}(0.115, -0.1),\ \text{and}\ \bar{D}_4^I = \mathrm{diag}(0.18, -0.15) \tag{52}$$

in Eq. (50). These four damped gyroscopically stabilized exponentially stable subsystems are

$$\bar{s}_1 = \{6.5, 5.25, \sqrt{35}, 3.53, -0.4, 0.43\},$$
$$\bar{s}_2 = \{4.5, 3.75, 5, 2.92, -0.15, 0.165\}$$
$$\bar{s}_3 = \{3, 2.25, \sqrt{15}, 4.55, -0.1, 0.115\},$$
$$\bar{s}_4 = \{1.75, 1.25, \sqrt{10}, 2.01, -0.15, 0.18\} \tag{53}$$

Thus, the 11-degree-of-freedom damped gyroscopically stabilized system described by

$$\ddot{u}_i + \bar{D}_i^I\dot{u}_i + \bar{G}_i\dot{u}_i + \Lambda_i u_i = 0,\ i = 1, 2, 3, 4 \tag{54}$$

$$\ddot{v} + D_s^{\mathrm{Diss}}\dot{v} \qquad + \Lambda_s v = 0 \tag{55}$$

with the matrices $\bar{G}_i, \bar{D}_i^I, i = 1, 2, 3, 4$, and D_s^{Diss} given above is guaranteed to be exponentially stable.

Simulations to computationally corroborate Result 1 (recall, $u_i = [x_{2i-1}, x_{2i}]^T$, $i = 1, 2, 3, 4$) with the initial conditions chosen as

$$x_{2i-1}(t) = 0.01,\ x_{2i}(t) = -0.02,\ \dot{x}_{2i-1}(t) = -0.02,$$
$$\dot{x}_{2i}(t) = 0.01,\ i = 1, 2, 3, 4 \tag{56}$$

are shown in Fig. 2. The responses $x_1(t), x_3(t), x_5(t)$, and $x_7(t)$ of each of the four damped gyroscopically stabilized subsystems in Eq. (53) show exponential decay. The response $v(t) = [x_9(t), x_{10}(t), x_{11}(t)]^T$ of the uncoupled three-degree-of-freedom dissipatively damped subsystems [see Eqs. (49) and (37)] whose equations are

$$\ddot{x}_9 + 0.01\dot{x}_9 + x_9 = 0,\ \ddot{x}_{10} + 0.03\dot{x}_{10} + 3x_{10} = 0,\ \text{and}$$
$$\ddot{x}_{11} + 0.06\dot{x}_{11} + 5x_{11} = 0 \tag{57}$$

are exponentially stable, and the response $v(t) = [x_9, x_{10}, x_{11}]^T$ is not shown for brevity. □

Remark 7: Because the elements along the diagonal of the matrix Λ in Eq. (19) can be ordered in any manner with no loss of generality (Remark 1), one could therefore choose to obtain n unstable potential subsystems, each with two degrees of freedom, by including different pairs of the negative eigenvalues in the set of n diagonal matrices $\{\Lambda_1, \Lambda_2, \ldots, \Lambda_n\}$ from among the totality of the $2n$ negative eigenvalues of the unstable potential matrix, instead of the ones used in Numerical Example 1, each negative eigenvalue appearing in only one of the n matrices.

In fact, it is easy to see that there are

$$Q = \prod_{i=1}^{n}[2n - (2i - 1)] = 1 \times 3 \times 5 \times \ldots \times (2n - 1) \tag{58}$$

ways of pairing the $2n$ distinct negative eigenvalues of Λ to form the n uncoupled two-degree-of-freedom unstable potential subsystems described by the matrices Λ_i, $i = 1, 2, \ldots, n$.

For example, when $n = 2$ and $K_{p-} = [-1, -2, -3, -4]$ we get $Q = 1 \times 3 = 3$ possible pairings of these four negative eigenvalues, so that 1) $\Lambda_1 = \mathrm{diag}(-2, -1), \Lambda_2 = \mathrm{diag}(-4, -3)$; 2) $\Lambda_1 = \mathrm{diag}(-3, -1), \Lambda_2 = \mathrm{diag}(-4, -2)$; and 3) $\Lambda_1 = \mathrm{diag}(-4, -1)$, $\Lambda_2 = \mathrm{diag}(-3, -2)$. When $n = 4$, as in Numerical Example 1,

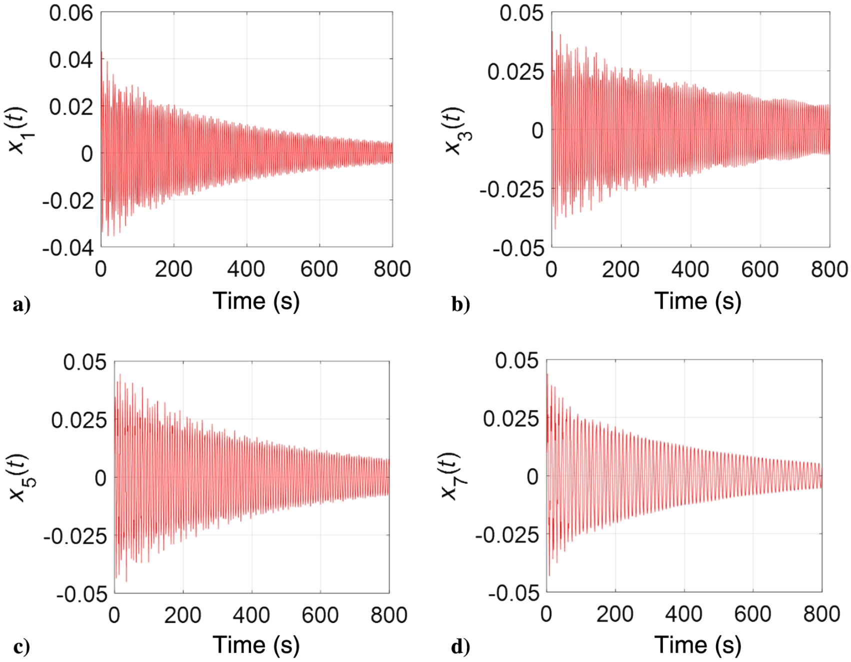

Fig. 2 Exponentially stable responses of gyroscopically stabilized subsystems $\bar{r}_i$ in Eq. (47): a) $i = 1$, b) $i = 2$, c) $i = 3$, and d) $i = 4$.

the number of such pairings exponentially explodes to $Q = 1 \times 3 \times 5 \times 7 = 105$!

Each such pairing of the $2n$ negative eigenvalues leads to a set of n two-degree-of-freedom (unstable) potential systems, which in turn lead to a set of n two-degree-of-freedom gyroscopic potential subsystems that have the structure described by Eq. (28). □

Numerical Example 2: As stated in Remark 7, one could re-order the negative eigenvalues of the matrix Λ contained in the row vector K_{p-} in a manner differently from that in Numerical Example 1, so that, for example, one pairing (out of the 105 possible parings) is

$$K_{p-} = [\overbrace{-6, -3}^{(-\lambda_1,-\lambda_2)}, \overbrace{-2.25, -5.25}^{(-\lambda_3,-\lambda_4)}, \overbrace{-4.5, -1.75}^{(-\lambda_5,-\lambda_6)}, \overbrace{-3.75, -1.25}^{(-\lambda_7,-\lambda_8)}] \quad (59)$$

instead of that in Eq. (41). The pairing of these eigenvalues is shown by square brackets above each pair, and the λ_i's are relabeled above each square bracket. This leads to four two-degree-of-freedom unstable potential subsystems:

$$\Lambda_1 = \text{diag}(-6, -3), \qquad \Lambda_2 = \text{diag}(-2.25, -5.25),$$
$$\Lambda_3 = \text{diag}(-4.5, -1.75), \text{ and } \Lambda_4 = \text{diag}(-3.75, -1.25) \quad (60)$$

instead of the ones use earlier in Eq. (42). We keep Λ_s unchanged, as in Eqs. (42). The subsystems Λ_1, Λ_3, and Λ_4 belong to Case 1 of Lemma 1. The subsystem Λ_2 belongs to Case 2 of Lemma 1, since $-\lambda_4 < -\lambda_3 < 0$.

Corresponding to the matrices in Eq. (60), the parameters $g_i, i = 1, 2, 3, 4$, in the matrices $G_i, i = 1, 2, 3, 4$, [see Eq. (43)] are chosen to satisfy Eq. (45) so that each of the four gyroscopic systems is marginally stable. They are

$$\bar{g}_1 = \sqrt{20} \approx 4.47, \quad \bar{g}_2 = \sqrt{20} \approx 4.47,$$
$$\bar{g}_3 = \sqrt{15} \approx 3.87, \text{ and } \bar{g}_4 = \sqrt{15} \approx 3.87 \quad (61)$$

These values of $\bar{g}_i, i = 1, 2, 3, 4$, when used in Eq. (43) yield the gyroscopic matrices $\bar{G}_i = \text{Skew}(\bar{g}_i), i = 1, 2, 3, 4$, which make the five decoupled systems

$$\ddot{u}_i + \bar{G}_i \dot{u}_i + \Lambda_i u_i = 0, \; i = 1, 2, \ldots n \quad (62)$$

$$\ddot{v} + \qquad \Lambda_s v = 0 \quad (63)$$

marginally stable. Instead of those shown in Eq. (47), the four gyroscopically stabilized subsystems are

$$\bar{r}_1 = \{6, 3, \sqrt{20}, 1.58\}, \quad \bar{r}_2 = \{2.25, 5.25, \sqrt{20}, 2.37\},$$
$$\bar{r}_3 = \{4.5, 1.75, \sqrt{15}, 1.77\}, \text{ and } \bar{r}_4 = \{3.75, 1.25, \sqrt{15}, 2.38\} \quad (64)$$

Note that the gyroscopic forces needed to gyro-stabilize these subsystems are on the whole smaller than those needed in Numerical Example 1 [see Eq. (46)]. This points out that proper pairing of the negative eigenvalues of Λ has a substantial effect on the magnitude of the gyroscopic forces needed to stabilize the unstable potential systems [compare Eqs. (41) and (59)]. Note that the first element of the ordered quadruple $\bar{r}_2$ is smaller in value than its second element; this affects the degree of freedom that receives positive feedback in this subsystem.

Next, we seek to make each of these four gyroscopically stabilized subsystems in Eq. (62) exponentially stable through the introduction of an appropriate linear indefinite damping matrix.

The subsystem described by $v(t)$ in Eq. (44), which has a positive definite potential matrix, can be made exponentially stable trivially as explained earlier in Remark 6, and we shall not bother with it.

We therefore focus attention on the responses $u_i, i = 1, 2, 3, 4$, of the four decoupled subsystems that are each gyroscopically stabilized. Our aim is to seek indefinite damping matrices $\bar{D}_i^I, i = 1, 2, 3, 4$, that guarantee the exponential stability of each of these subsystems, thus making the entire system shown in Eq. (48) exponentially stable.

This is done by first obtaining the four different regions of exponential stability in the $\alpha - d$ plane for each of these four gyroscopically stabilized subsystems described in Eq. (64) by the quadruples $\bar{r}_i, i = 1, 2, 3, 4$. Any point defined by the coordinates $(\bar{\alpha}_i, \bar{d}_i)$ that lies inside the region of exponential stability of the ith damped gyroscopically stabilized subsystem, $1 \le i \le 4$, can then be used to guarantee exponential stability of the ith subsystem, i.e., exponential stability of $u_i(t)$.

The regions of exponential stability for the indefinitely damped four gyroscopically stabilized, two-degree-of-freedom subsystems $\bar{r}_i, i = 1, 2, 3, 4$, shown in Eq. (64) are found as explained in Appendix A. They are shown in Fig. 3a. Picking any point $(\bar{\alpha}_i, \bar{d}_i)$ in the

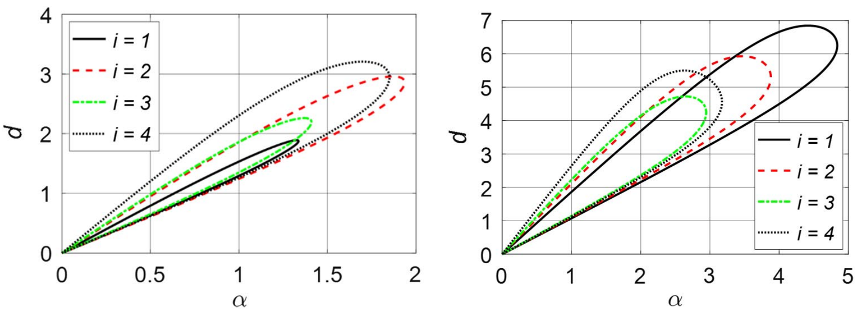

Fig. 3 Exponential stability regions for the four subsystems with quadruples $\bar{r}_i$, $i = 1, 2, 3, 4$, in a) Eq. (64) and b) Eq. (66).

region of stability of the ith gyroscopically stabilized subsystem and setting $\bar{D}_i^I = \mathrm{diag}(\bar{d}_i, -\bar{\alpha}_i)$, $i = 1, 3, 4$, and $\bar{D}_2^I = (-\bar{\alpha}_2, \bar{d}_2)$ will render the gyroscopically stabilized subsystems of Eq. (64) exponentially stable. Some care is required with $\bar{D}_2^I$ for the two-degree-of-freedom subsystem formed from Λ_2 in Eq. (64) for the coordinate $u_2(t)$, which uses Case 2 of Lemma 1. For brevity, the response of each damped subsystem when a point in its corresponding stability region is chosen to specify the indefinite damping matrices $\bar{D}_i^I$, $i = 1, 2, 3, 4$, is not shown.

To illustrate the effect of the magnitude of the gyroscopic forces applied to each of the four subsystems on their respective regions of exponential stability, we increase the gyroscopic forces described by $\bar{g}_i$, $i = 1, 2, 3, 4$ from those shown in Eq. (61) to

$$\bar{g}_1 = 7,\ \bar{g}_2 = 6,\ \bar{g}_3 = 5,\ \text{and}\ \bar{g}_4 = 5 \tag{65}$$

so that instead of Eq. (64), we now have

$$\bar{r}_1 = \{6, 3, 7, 5.61\}, \qquad \bar{r}_2 = \{2.25, 5.25, 6, 4.65\},$$
$$\bar{r}_3 = \{4.5, 1.75, 5, 3.62\},\ \text{and}\ \bar{r}_4 = \{3.75, 1.25, 5, 3.96\} \tag{66}$$

The first two parameters of each quadruple in Eq. (66) are the same as those in Eq. (64), and therefore Λ_i, $i = 1, 2, 3, 4$, are same as before [Eq. (60)]; that is, it is same unstable potential system Λ we had before. The fan-shaped stability regions that result using Eq. (66) are shown in Fig. 3b. The stability loops are seen to be somewhat bigger and fatter.

In practical situations one would usually desire larger exponential stability regions because they bestow greater robustness. The increased robustness appears to come with increased investment in gyroscopic forces used for gyroscopic stabilization though. □

We provide below a general methodology for guaranteeing asymptotically stable behavior of an N-degree-of-freedom potential system that has $2n$ distinct negative eigenvalues and $(N-2n)$ positive eigenvalues. The methodology comprises a setup stage and a four-step procedure.

Result 2: Consider the N-degree-of-freedom unstable potential system

$$\ddot{x} + \Lambda x = 0 \tag{67}$$

where Λ is a diagonal matrix with the eigenvalues of $\hat{K} = M^{-1/2}\tilde{K}M^{-1/2}$ along its diagonal. Assume that the potential matrix $\hat{K}(\tilde{K})$ has $2n \le N$ negative eigenvalues that are distinct, and $N - 2n$ positive eigenvalues.

Setup: Put the $2n$ negative eigenvalues of $\hat{K}$ in a row vector, K'_{p-}, placing them in any desired order; place the positive eigenvalues of $\hat{K}$ in the row vector K_{p+} in *any* desired order. The vector K'_{p-} and the vector K_{p+} so obtained are

$$K'_{p-} = [\overline{-\lambda_1, -\lambda_2}, \overline{-\lambda_3, -\lambda_4}, \ldots, \overline{-\lambda_{2n-1}, -\lambda_{2n}}]\ \text{and}$$
$$K_{p+} = [\lambda_{2n+1}, \lambda_{2n+2}, \ldots, \lambda_N],\ \lambda_i > 0,\ i = 1, 2, \ldots N \tag{68}$$

The negative eigenvalues listed in K'_{p-} are paired as shown by the square bracket above each pair. It is convenient to have the first member of each pair less that its second member. If the first member in any pair is larger than its second, the positions of the two members in the pair are switched, without loss of generality. We shall assume that this has been done so that we obtain the new $2n$-row vector with

$$K'_{p-} = [\overline{-\lambda_1, -\lambda_2}, \overline{-\lambda_3, -\lambda_4}, \ldots \overline{-\lambda_{2n-1}, -\lambda_{2n}}],$$
$$\text{with}\ -\lambda_{2i-1} < -\lambda_{2i} < 0,\ i = 1, 2, \ldots, n \tag{69}$$

in which we again denote the pairs (after relabeling the λ's, as needed) by square brackets above each pair. In each pair in Eq. (69) the member on the left is less than the member on its right.

The set of equations in Eq. (67) are uncoupled, and with no loss of generality, we can rewrite Eq. (66) (with a possible relabeling of the x_i's) so that

$$\ddot{x} + \Lambda x = 0, \text{where now}\ \Lambda = \mathrm{diag}(\mathrm{K}_{p-}, K_{p+}) \tag{70}$$

In what follows we assume that this relabeling is done, and Eq. (70) represents our unstable potential system.

After this setup stage, which basically organizes the eigenvalues of the unstable potential matrix $\hat{K}$ in a systematic way, we now proceed with the four-step methodology.

Step 1: a) Using the row vector K_{p-} [Eq. (69)], to form the 2-by-2 unstable potential matrices $\Lambda_i = \mathrm{diag}(-\lambda_{2i-1}, -\lambda_{2i}), i = 1, 2, \ldots, n$, and the 2-by-2 unstable potential systems

$$\ddot{u}_i + \Lambda_i u_i = 0,\ i = 1, 2, \ldots, n \tag{71}$$

in which $u_i = [x_{2i-1}, x_{2i}]$, $i = 1, 2, \ldots, n$, and the x_i's come from Eq. (70). Notice that in the setup stage [Eq. (69)] we have ensured that in each of the unstable potential matrices Λ_i, $1 \le i \le n$, we have $-\lambda_{2i-1} < -\lambda_{2i}$.

b) Specify the remainder of the potential system, which is marginally stable, as

$$\ddot{v} + \Lambda_s v = 0 \tag{72}$$

in which $v = [x_{2n+1}, x_{2n+2}, \ldots, x_N]^T$, and $\Lambda_s = \mathrm{diag}(K_{p+})$.

Thus the unstable part of the potential system is organized as a set of n uncoupled two-degree-of-freedom unstable (potential) systems, and the stable part of the potential system is organized into an $(N-2n)$-degree-of-freedom marginally stable system. The potential matrix of the unstable potential system is thus represented now as $\Lambda = \mathrm{BLKdiag}(\Lambda_1, \Lambda_2, \ldots, \Lambda_n, \Lambda_s) = \mathrm{BLKdiag}(\Lambda_u, \Lambda_s)$.

Step 2: Choose $\bar{\delta}_i$, $i = 1, 2, \ldots, n$, such that

$$\bar{g}_i = \sqrt{-\mathrm{Trace}(\Lambda_i) + 2\sqrt{Det(\Lambda_i)} + \bar{\delta}_i^2}, \quad \bar{\delta} \neq 0,\ i = 1, 2, \ldots, n \tag{73}$$

The n unstable potential systems

$$\ddot{u}_i + \begin{bmatrix} 0 & \bar{g} \\ -\bar{g}_i & 0 \end{bmatrix} \dot{u}_i + \begin{bmatrix} -\lambda_{2i-1} & 0 \\ 0 & -\lambda_{2i} \end{bmatrix} u_i = 0,$$
$$-\lambda_{2i-1} < -\lambda_{2i} < 0, \; i = 1, 2, \ldots, n \tag{74}$$

are then gyroscopically stabilized. They are denoted by $\bar{r}_i = \{\lambda_{2i-1}, \lambda_{2i}, \bar{g}_i, \bar{\delta}_i\}$, $i = 1, 2, \ldots, n$.

Step 3: Consider the n damped systems

$$\ddot{u}_i + \underbrace{\begin{bmatrix} \bar{d}_i & 0 \\ 0_i & -\bar{\alpha}_i \end{bmatrix}}_{\bar{D}_i^I} \dot{u}_i + \underbrace{\begin{bmatrix} 0 & \bar{g}_i \\ -\bar{g}_i & 0 \end{bmatrix}}_{\bar{G}_i} \dot{u}_i + \begin{bmatrix} -\lambda_{2i-1} & 0 \\ 0 & -\lambda_{2i} \end{bmatrix} u_i = 0,$$
$$-\lambda_{2i-1} < -\lambda_{2i} < 0, \; i = 1, 2, \ldots, n, \; d_i, \alpha_i > 0 \tag{75}$$

also denoted by $s_i = \{\lambda_{2i-1}, \lambda_{2i}, \bar{g}_i, \bar{\delta}_i, -\bar{\alpha}_i, \bar{d}_i\}$, $i = 1, 2, \ldots, n$.

To obtain the appropriate values of $\bar{\alpha}_i, \bar{d}_i > 0$ to guarantee exponential stability, do the following.

1) Find the region of exponential stability in the $\alpha - d$ plane for each of the n gyroscopically stabilized subsystems $\bar{r}_i$, $i = 1, 2, \ldots, n$ [Eq. (74)]. Use Case 1 from Appendix A to do this, since $-\lambda_{2i-1} < -\lambda_{2i} < 0$, $i = 1, 2, \ldots, n$.

2) Pick any point with coordinates $(\bar{\alpha}_i, \bar{d}_i)$ that lies in the stability zone of the ith subsystem, $1 \le i \le n$, and thereby obtain the parameters $\bar{\alpha}_i$ and $\bar{d}_i$ in Eq. (75).

Step 4: Add dissipative damping $D_s^{\text{Diss}} > 0$ to the subsystem in Eq. (72)

$$\ddot{v} + D_s^{\text{Diss}} \dot{v} + \Lambda_s v = 0 \tag{76}$$

where D_s^{Diss} is any $(N - 2n)$-by-$(N - 2n)$ positive definite matrix.

The corresponding matrices that make Eq. (9) (with $D = \bar{D}$) exponentially stable are then $\bar{G} \coloneqq G = \text{BLKdiag}(\bar{G}_1, \bar{G}_2, \ldots, \bar{G}_n, 0)$, $\bar{D} = \text{BLKdiag}(\bar{D}_1^I, \bar{D}_2^I, \ldots \bar{D}_n^I, D_s^{\text{Diss}})$. This result directly translates into making the unstable gyroscopically stabilized potential system in Eq. (1) exponentially stable (see Remark 3). □

B. Multiple Negative Eigenvalues of the Potential Matrix $\hat{K}$

We show in this subsection that though a two-degree-of-freedom gyroscopically stabilized system with identical eigenvalues cannot be made exponentially stable by using indefinite damping, an N-degree-of-freedom system with several negative eigenvalues with different multiplicities greater than 1 can almost always be made exponentially stable. The necessary and sufficient conditions that permit exponential stability are obtained.

Consider the diagonal potential matrix $\Lambda = BLK(\Lambda_u, \Lambda_s) \coloneqq \text{BLKdiag}(\Lambda_1, \Lambda_2, \ldots, \Lambda_n, \Lambda_s)$ with n 2-by-2 diagonal potential matrices $\Lambda_i < 0$, $i = 1, 2 \ldots, n$, and a square $(N - 2n)$ diagonal matrix $\Lambda_s > 0$. If the two negative diagonal elements in each Λ_i are different, then each of these n unstable potential subsystems, after gyroscopic stabilization, is guaranteed to be made exponentially stable by use of an appropriate indefinite damping matrix (Lemma 1). If, however, even one potential subsystem, say, $\Lambda_k < 0$, $1 \le k \le n$, has identical elements, then after gyroscopic stabilization, that system cannot be made exponentially stable. And hence the entire system cannot be made exponentially stable.

The unstable potential subsystems Λ_i, $i = 1, 2 \ldots, n$, are, of course, obtained by organizing the $2n$ negative eigenvalues of the potential system into n pairs, with each pair forming the diagonal elements of these n diagonal matrices (see the setup stage in Result 2).

Remark 7: Consider an 11-degree-of-freedom potential system, but assume that now the eigenvalues of the system are

$$\Lambda = \text{diag}(-6, -5.25, -4.5, -3.75, -3, -1.25, -1.25, -1.25, 1, 3, 5) \tag{77}$$

so that we have a single negative eigenvalue, -1.25, that has a multiplicity of 3. A gyroscopically stabilized two-degree-of-freedom subsystem that has a potential matrix diag$(-1.25, -1.25)$ cannot be made, after gyroscopic stabilization, exponentially stable by using an indefinite damping matrix.

But with no loss of generality, the eight negative eigenvalues of the potential matrix Λ in Eq. (77), which constitute the row vector K_{p-}, can be listed in any order (Remark 1), and therefore can be paired to create two-degree-of-freedom (unstable) potential subsystems in any way we want. The negative eigenvalues shown in Eq. (77) can be listed, for example, as

$$K_{p-} = [\overbrace{-6, -3}^{(-\lambda_1, -\lambda_2)}, \overbrace{-5.25, -1.25}^{(-\lambda_3, -\lambda_4)}, \overbrace{-4.5, -1.25}^{(-\lambda_5, -\lambda_6)}, \overbrace{-3.75, -1.25}^{(-\lambda_7, -\lambda_8)}] \tag{78}$$

and paired, as shown by the brackets above each pair. Each pair has two eigenvalues that differ from one another. The pairing in Eq. (78) can be used for describing the potential matrices $\Lambda_i = \text{diag}(-\lambda_{2i-1}, -\lambda_{2i})$, $i = 1, 2, 3, 4$, so that

$$\Lambda_1 = \text{diag}(-6, -3), \qquad \Lambda_2 = \text{diag}(-5.25, -1.25),$$
$$\Lambda_3 = \text{diag}(-4.5, -1.25), \text{ and } \Lambda_4 = \text{diag}(-3.75, -1.25) \tag{79}$$

As seen in Eq. (79), the multiple negative eigenvalue -1.25 of Λ is simply distributed among three pairs and the pairing leads to four two-degree-of-freedom unstable (uncoupled) potential subsystems. Each Λ_i, $i = 1, \ldots, 4$, has distinct elements; exponential stability of the potential system is then guaranteed after gyroscopic stabilization (Result 2). □

Remark 7 shows that it may be possible, at times, to circumvent the problem raised by multiple negative eigenvalues of the potential matrix $\hat{K}$. The question is whether it is always possible to have such a pairing of the $2n$ negative eigenvalues of a potential system so that each of the n pairs has distinct elements in it.

It is clear that there can be no such guarantee on being able to do this as is obvious when the $2n$ negative eigenvalues of the potential system are identical. But the situation can get even subtler. To illustrate, suppose that an 11-degree-of-freedom unstable potential system has the eigenvalues

$$\Lambda = \text{diag}(-6, -5.25, -4.5, -1.25, -1.25, -1.25, -1.25, -1.25, 1, 3, 5) \tag{80}$$

instead of those in Eq. (77). Now the multiplicity of the eigenvalue, -1.25, is 5 and no pairing of the 8 negative eigenvalues can be made with each pair having different eigenvalues in it; at least one of the 2-by-2 matrices $\Lambda_i < 0$, $i = 1, 2, 3, 4$, that contain the 8 negative eigenvalues of Λ will have identical elements. On the other hand, if the multiplicity of the eigenvalue -1.25 is 4, then such a pairing is possible, and one can make four two-degree-of-freedom (unstable) potential subsystems that can then be gyroscopically stabilized, and thereafter made exponentially stable!

An additional difficulty that may arise in N-degree-of-freedom unstable potential systems in which N could be in the thousands is that there may be many negative eigenvalues, each with different multiplicities greater than 1. Under what conditions are we guaranteed to find a pairing of the $2n$ negative eigenvalues such that each pair has different eigenvalues in it? That is, under what conditions can each of the n two-degree-of-freedom unstable potential subsystems have distinct negative eigenvalues, so that after gyroscopic stabilization each of them is guaranteed to be made exponentially stable by using an indefinite damping matrix? The answer to this question takes us to the field of combinatorics. In order not to break the chain of thought, these questions are answered in some generality in Results A1 and A2 in Appendix B and are stated here.

Lemma 2: Consider the $2n$ negative eigenvalues of the symmetric matrix $\hat{K}$. Let k of these $2n$ eigenvalues be distinct and denote the row vector that contains these (negative) distinct eigenvalues by $\bar{K}_{p-} = [-\lambda_1, -\lambda_2, \ldots, -\lambda_k]$, with $\lambda_i > 0$, $i = 1, 2, \ldots k$. Denote the multiplicity of the negative eigenvalue $-\lambda_i$, by $m_i \ge 1$, $i = 1, 2, \ldots, k$.

Let the distinct eigenvalue $-\lambda_l$, $1 \le l \le k$, have multiplicity m_l, such that $m_l \ge m_i, \forall\ i \ne k$. Then a necessary and sufficient condition that n pairs can be made from these $2n$ negative eigenvalues so that no pair has identical eigenvalues in it is that

$$m_l \le n \tag{81}$$

Proof: Note that in the statement of the lemma, no negative eigenvalue of Λ has a multiplicity that exceeds m_l. The proof is given in Result A1 in Appendix B. □

We illustrate this lemma with the following numerical example.

Numerical Example 3: 1) Let the six negative eigenvalues of $\hat{K}$ be $K_{p-} = [-2, -3, -2, -3, -4, -5]$, and let $K_{p+} = [1, 3]$. Here $N = 8, n = 3$, and $\bar{K}_{p-} = [-2, -3, -4, -5]$ so that $k = 4$. The eigenvalue -2 has multiplicity $m_l = 2$, which is not exceeded by any of the other negative eigenvalues; also, $m_l \le n$. Hence, one can pair the six eigenvalues so that each pair has distinct numbers in it. For example, by reordering the elements of K_{p-} one can write $K_{p-} = [\overbrace{-3, -2}, \overbrace{-3, -2}, \overbrace{-5, -4}]$ in which the square brackets show the pairs. The three two-degree-of-freedom unstable uncoupled potential systems obtained from this pairing are

$$\Lambda_1 = \mathrm{diag}(-3, -2), \quad \Lambda_2 = \mathrm{diag}(-3, -2), \quad \text{and } \Lambda_3 = \mathrm{diag}(-5, -4) \tag{82}$$

and Lemma 1 guarantees that these gyroscopically stabilized potential subsystems can be made exponentially stable.

2) Let the six negative eigenvalues of $\hat{K}$ be, $K_{p-} = [-2, -2, -2, -2, -3, -5]$, and let $K_{p+} = [1, 3]$. Here $N = 8$, and $n = 3$. The negative eigen-value -2 has multiplicity 4 and no negative eigenvalue has a higher multiplicity. Since $m_l = 4 > n$ there can be no pairing of the negative eigenvalues so that each pair has different numbers in it. For example, the pairing shown by $K_{p-} = [\overbrace{-3, -2}, \overbrace{-5, -2}, \overbrace{-2, -2}]$ leads to the smallest number of pairs (namely, 1, in this case) that have identical negative eigenvalues in them. An alternative pairing $K_{p-} = [\overbrace{-3, -5}, \overbrace{-2, -2}, \overbrace{-2, -2}]$ has a larger number of such pairs, namely, two pairs now. The first pairing leads to the potential submatrices:

$$\Lambda_1 = \mathrm{diag}(-3, -2),\ \Lambda_2 = \mathrm{diag}(-5, -2),\ \text{and}\ \Lambda_3 = \mathrm{diag}(-2, -2) \tag{83}$$

Thus, the two-degree-of-freedom potential subsystem described by the potential matrix Λ_3 has identical eigenvalues. Though it can be gyroscopically stabilized, it cannot be made exponentially stable through the use of an indefinite damping matrix. By Lemma 1, this potential subsystem can only be made marginally stable; hence, the entire system remains only marginally stable. □

Lemma 2 leads to the following result.

Result 3: Consider the N-degree-of-freedom unstable potential system

$$\ddot{x} = \Lambda x \tag{84}$$

where Λ is a diagonal matrix with the eigenvalues of $\hat{K}$ running down its diagonal. The matrix $\hat{K}$ has $2n \le N$ negative eigenvalues and $(N - 2n)$ positive eigenvalues. Let one or more of the negative eigenvalues of $\hat{K}$ have multiplicity greater than 1. Let m_l be the highest multiplicity among the distinct negative eigenvalues of $\hat{K}$. If

$$m_l \le n \tag{85}$$

the N-degree-of-freedom unstable potential system can be gyroscopically stabilized, and it can always be made exponentially stable by using (an) indefinite damping (matrix). We shall refer to such an unstable potential system that satisfies relation (85) as being *generic*.

Alternatively stated, every generic MDOF unstable potential system that has an even number of negative eigenvalues can be gyroscopically stabilized, and the gyroscopically stabilized system can always be made exponentially stable.

Proof: By Lemma 2, the negative eigenvalues in K_{p-} can always be paired so that the negative eigenvalues in each of the 2-by-2 matrices Λ_i, $i = 1, 2, \ldots, n$, are different. Result 2 then guarantees that the gyroscopically stabilized system can always be made exponentially stable by using indefinite damping. □

Remark 9: Consider an unstable 100-degree-of-freedom potential system whose potential matrix has, say, 30 negative eigenvalues. It is generic if none of these negative eigenvalues have a multiplicity that exceeds 15. Although it might be possible for a physical system to have a negative eigenvalue whose multiplicity may perhaps be 3 or 4, from a practical standpoint it is very unlikely to have a negative eigenvalue whose multiplicity exceeds 15, and therefore this 100-degree-of-freedom unstable potential system is more than likely to be generic, which implies that it can be made exponentially stable after gyroscopic stabilization by using indefinite damping. □

Remark 10: In Lemma 2, when the $2n$ negative eigenvalues of $\hat{K}$ have multiplicities and $m_l \le n$, the number of ways of pairing these negative eigenvalues so that each of the n pairs has different eigenvalues in it does not seem to be easy to find in general, and appears to be not explicitly known. This is unlike the case when the $2n$ negative eigenvalues of $\hat{K}$ are distinct [see Eq. (58)]. □

Remark 11: When the multiplicity of the negative eigenvalues of the unstable potential matrix Λ_u exceeds n, there is a negative eigenvalue whose multiplicity m_l is the highest, with $m_l = n + r$. From Lemma 2, the $2n$ negative eigenvalues of the potential matrix cannot be paired now so that each pair has different eigenvalues in it, and there will be some pairs containing identical eigenvalues. Let the number of pairs with identical eigenvalues be I_r. Result A2 in Appendix B shows that there exists a pairing of the $2n$ negative eigenvalues for which I_r is a minimum, and that this minimum value of I_r equals r. A constructive way of obtaining a pairing among the $2n$ negative eigenvalues that has a minimum value of I_r is also given in Result A2. □

This means that when the unstable potential system is nongeneric there will be at least r 2-by-2 unstable potential subsystems that will have identical negative eigenvalues. Thus, there will be a minimum of r 2-by-2 unstable potential subsystems among the $\Lambda_i < 0, = 1, 2, \ldots, n$, which, after gyroscopic stabilization, cannot be made exponentially stable by using indefinite damping matrices and will remain marginally stable. Such a nongeneric MDOF system, cannot be made exponentially stable through the use of indefinite damping using the methodology developed; it will only be marginally stable.

III. Conclusions

This paper shows that a generic unstable potential MDOF system whose potential matrix has an even number of negative eigenvalues can always be gyroscopically stabilized, and the gyroscopically stabilized MDOF system can further always be made *exponentially stable* by using an uncountably infinite number of indefinite damping matrices. By "generic" it is meant that the unstable potential matrix of the system has the following property: it does not have any negative eigenvalue whose multiplicity exceeds half the total number of its negative eigenvalues. Most large-scale, real-life gyroscopically stabilized systems that have many degrees of freedom would therefore be encompassed by this property that defines genericity, and therefore would be guaranteed to be made exponentially stable.

Besides showing that this result is true for unstable conservative MDOF systems described above, a general stepwise constructive, simple methodology of achieving exponential stability for any generic MDOF system is provided. The methodology provides a practical way of achieving exponential stability of gyroscopically stabilized unstable potential systems in real-world applications through the simultaneous use of positive and negative velocity feedback.

Making the gyroscopically stabilized system exponentially stable has yet another important consequence. Many aerospace and mechanical systems are nonlinear, and the linear equations dealt with in this paper are often the result of linearizations around equilibrium points of these nonlinear systems. Because gyroscopic stabilization leads to marginal stability of an unstable potential (conservative) system, there is no guarantee that the original nonlinear system from which the linearized system is obtained will remain stable at this equilibrium point, because the equilibrium point of the linearized system is nonhyperbolic. The introduction of exponential stability in a gyroscopically stabilized system through indefinite damping makes the equilibrium point of the linearized system hyperbolic, and thus ensures that in the vicinity of this equilibrium point the original nonlinear system remains stable.

The 150-year-old KTC result is an important paradigm and a cornerstone of the theory of linear stability. As in this paper, the KTC theorem deals solely with systems subjected to three qualitatively different forces: potential positional forces that make the system unstable, called unstable potential systems for short; gyroscopic forces; and linear-in-velocity damping forces characterized by a damping matrix. It states that any unstable conservative MDOF system that can be gyroscopically stabilized becomes unstable in the presence of a linear-in-velocity damping force that is dissipative and characterized by a positive definite damping matrix [1–6]. In contrast to the KTC result, we have shown here that generic unstable conservative MDOF systems can be gyroscopically stabilized and can always be made exponentially stable, by the addition of a linear-in-velocity damping force that is indefinite and characterized by an indefinite damping matrix. The approach provides the entire stability boundaries explicitly in a straightforward manner. The results presented in this paper therefore point to a new and different paradigm from KCT when dealing with linear-in-velocity indefinitely damped gyroscopically stabilized systems.

The stability of such gyroscopically stabilized systems is brought about by the simultaneous use of both negative velocity feedback (or dissipative damping) and positive velocity feedback. It is shown that these competing feedbacks, when appropriately devised and combined, constructively interact with each other bestowing guaranteed exponential stability. At root, the basic idea in achieving such exponential stability is the balanced simultaneous dissipation and injection [16,17] of energy to the gyroscopically stabilized system. Perhaps Kelvin and Tait did not think of injecting and simultaneously dissipating energy, because the injection of energy to bring about exponential stability is somewhat counterintuitive.

The methodology developed in the paper provides a guaranteed approach to make gyroscopically stabilized spinning aerospace and mechanical systems exponentially stable, something thus far thought impossible to do in the presence of energy dissipation.

Appendix A: Explicit Determination of the Exponential Region of Stability for a Gyroscopically Stabilized Two-Degree-of-Freedom Potential System

Consider a given gyroscopically stabilized dynamic system

$$\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix}+\underbrace{\begin{bmatrix}0 & \bar{g}\\ -\bar{g} & 0\end{bmatrix}}_{\bar{G}}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}+\underbrace{\begin{bmatrix}-\lambda_1 & 0\\ 0 & -\lambda_2\end{bmatrix}}_{\Lambda}\begin{bmatrix}x_1\\ x_2\end{bmatrix},\quad \lambda_1,\lambda_2>0 \tag{A1}$$

in which

$$\bar{g}=\sqrt{-\mathrm{Trace}(\Lambda)+2\sqrt{\mathrm{Det}(\Lambda)}+\bar{\delta}^2},\qquad \bar{\delta}\neq 0$$

System (A1) is also denoted by the ordered quadruple $\bar{r}=\{\lambda_1,\lambda_2,\bar{g},\bar{\delta}\}$ in which the negative sign on the λ's is dropped.

Our objective is to introduce an indefinite damping matrix D^I so that system (A1) is exponentially stable.

We begin by specifying some notation. Consider the system

$$\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix}+\underbrace{\begin{bmatrix}d & 0\\ 0 & -\alpha\end{bmatrix}}_{D^I}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}+\underbrace{\begin{bmatrix}0 & \bar{g}\\ -\bar{g} & 0\end{bmatrix}}_{\bar{G}}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}+\underbrace{\begin{bmatrix}-\lambda_1 & 0\\ 0 & -\lambda_2\end{bmatrix}}_{\Lambda}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=0,\quad d,\alpha>0 \tag{A2}$$

We denote it by the ordered sextuple $s=\{\lambda_1,\lambda_2,\bar{g},\bar{\delta},-\alpha,d\}$. For convenience, we again drop the negative signs on the λ's. The (1,1) and (2,2) elements of D^I in Eq. (A2) are the sixth and fifth elements, respectively, of the ordered sextuple s. Similarly, the (1,1) and (2,2) elements of Λ without the minus signs are the first and the second elements, respectively, of s.

The sextuple $s^*=\{\lambda_2,\lambda_1,\bar{g},\bar{\delta},-\alpha,d\}$, which has its first two elements switched when compared with those in the sextuple s, therefore is the system

$$\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix}+\underbrace{\begin{bmatrix}d & 0\\ 0 & -\alpha\end{bmatrix}}_{D^I}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}+\underbrace{\begin{bmatrix}0 & \bar{g}\\ -\bar{g} & 0\end{bmatrix}}_{\bar{G}}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}+\underbrace{\begin{bmatrix}-\lambda_2 & 0\\ 0 & -\lambda_1\end{bmatrix}}_{\Lambda}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=0,\quad d,\alpha>0 \tag{A3}$$

Similarly, the ordered sextuple $s'=\{\lambda_1,\lambda_2,\bar{g},\bar{\delta},d,-\alpha\}$, which has its last two elements switched when compared with those in s, is the system

$$\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix}+\underbrace{\begin{bmatrix}-\alpha & 0\\ 0 & d\end{bmatrix}}_{D^I}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}+\underbrace{\begin{bmatrix}0 & \bar{g}\\ -\bar{g} & 0\end{bmatrix}}_{\bar{G}}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}+\underbrace{\begin{bmatrix}-\lambda_1 & 0\\ 0 & -\lambda_2\end{bmatrix}}_{\Lambda}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=0,\quad d,\alpha>0 \tag{A4}$$

We now consider three cases as in Lemma 1.

Case 1: The gyroscopically stabilized system described by the ordered quadruple $\bar{r}=\{\lambda_1,\lambda_2,\bar{g},\bar{\delta}\}$ has $-\lambda_1<-\lambda_2<0$.

We consider the damped system $s=\{\lambda_1,\lambda_2,\bar{g},\bar{\delta},-\alpha,d\}$, in which the first four elements of the sextuple are identical to those of $\bar{r}$. The first element, λ_1, of s is therefore greater than its second element λ_2, since $-\lambda_1<-\lambda_2<0$.

Our aim is to find an appropriate indefinite damping matrix $D^I=\mathrm{diag}(d,-\alpha)$ so that the response of system (A2) is exponentially stable. This entails finding one or more appropriate pairs of values $\{\alpha=\bar{\alpha},d=\bar{d}\}i=1,2,\ldots,$ so that this equation, (A2), is exponentially stable.

To find such pairs, the region of exponential stability is found in the $\alpha-d$ plane for the system in Eq. (A2) and denoted by $s=\{\lambda_1,\lambda_2,\bar{g},\bar{\delta},-\alpha,d\}$ in which the first element of s is greater than its second. The determination of this region is done as follows [5].

We define $k:=\dfrac{\lambda_1}{\lambda_2}>1$, and consider a ray starting from the origin of the $\alpha-d$ plane that has slope

$$u(\gamma):=\left[1+\frac{(k-1)}{\gamma}\right] \tag{A5}$$

so that this ray is described by the equation

$$d=u(\gamma)\alpha \tag{A6}$$

Such a ray that emanates from the origin O is shown in Fig. A1 by the dashed line.

We consider all such rays in the first quadrant of the $\alpha-d$ plane that go through the origin O and have slopes $u(\gamma_2)<u(\gamma)<u(\gamma_1)$, where

$$\gamma_1=1+\frac{1}{\lambda_2}\left[\frac{1}{2}\bar{\delta}^2+\sqrt{\lambda_1\lambda_2}-\frac{1}{2}\sqrt{\bar{\delta}^4+4\bar{\delta}^2\sqrt{\lambda_1\lambda_2}}\right] \tag{A7}$$

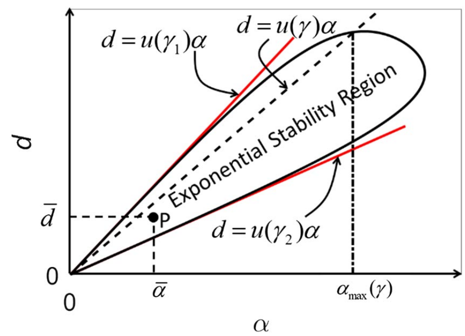

Fig. A1 Exponential stability region for system $s = \{\lambda_1, \lambda_2, \bar{g}, \bar{\delta}, -\alpha, \delta\}$, when $-\lambda_1 < -\lambda_2 < 0$.

and

$$\gamma_2 = 1 + \frac{1}{\lambda_2}\left[\frac{1}{2}\bar{\delta}^2 + \sqrt{\lambda_1\lambda_2} + \frac{1}{2}\sqrt{\bar{\delta}^4 + 4\bar{\delta}^2\sqrt{\lambda_1\lambda_2}}\right] \tag{A8}$$

The rays with these slopes are shown by the solid (red) lines in Fig. A1.

Every point along any such ray with slope $u(\gamma_2) < u(\gamma) < u(\gamma_1)$, like the one shown in Fig. A1 by the dashed line, whose α-coordinate satisfies the inequality

$$0 < \alpha < \sqrt{\frac{\gamma n(\gamma)}{\lambda_2(\gamma - 1)(k + \gamma + 1)}} := \alpha_{\max}(\gamma) \tag{A9}$$

where

$$n(\gamma) = \lambda_2[\bar{\delta}^2 + 2\sqrt{\lambda_1\lambda_2}](\gamma - 1) - \lambda_1\lambda_2 - \lambda_2^2(\gamma - 1) \tag{A10}$$

lies inside the region of exponential stability of the system. This stability region is outlined by the black, fan-shaped loop in Fig. A1.

Relations (A5–A10) give the explicit determination of this region. We can denote by S this exponential stability region that is so obtained for the system $s = \{\lambda_1, \lambda_2, \bar{g}, \bar{\delta}, -\alpha, \delta\}$, in which $\lambda_1 > \lambda_2 > 0$ [see Eq. (A2)].

Any representative point P with coordinates $\{\alpha = \bar{\alpha}, d = \bar{d}\}$ that lies inside S (see Fig. A1) when used in Eq. (A2) will make the gyroscopically stabilized system exponentially stable provided that $\lambda_1 > \lambda_2 > 0$. That is, the ordered sextuple $\bar{s} = \{\lambda_1, \lambda_2, \bar{g}, \bar{\delta}, -\bar{\alpha}, \bar{d}\}$ guarantees asymptotically stability. Because the stability region is connected and continuous [5], one obtains an uncountably infinite number of indefinite damping matrices that will make the system s [or, alternately Eq. (A2)] exponentially stable.

Thus, for the gyroscopically stabilized system $\bar{r} = \{\lambda_1, \lambda_2, \bar{g}, \bar{\delta}\}$ with $-\lambda_1 < -\lambda_2 < 0$, the system

$$\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix} + \underbrace{\begin{bmatrix}\bar{d} & 0\\ 0 & -\bar{\alpha}\end{bmatrix}}_{\bar{D}^I}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} + \underbrace{\begin{bmatrix}0 & \bar{g}\\ -\bar{g} & 0\end{bmatrix}}_{\bar{G}}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} + \underbrace{\begin{bmatrix}-\lambda_1 & 0\\ 0 & -\lambda_2\end{bmatrix}}_{\Lambda}\begin{bmatrix}x_1\\ x_2\end{bmatrix},$$
$$-\lambda_1 < -\lambda_2 < 0,\ d, \alpha > 0 \tag{A11}$$

is exponentially stable for any $(\bar{\alpha}, \bar{d})$ belonging to the region of exponential stability in the $\alpha - d$ plane obtained by using Eqs. (A5–A10).

It is important to note that the procedure given by Eqs. (A5–A10) to explicitly obtain the region of exponential stability requires that the sextuple s must have its first argument, λ_1, greater than its second, λ_2. Result 2 relies on this.

Case 2: The gyroscopically stabilized system described by the ordered quadruple $\bar{r} = \{\lambda_1, \lambda_2, \bar{g}, \bar{\delta}\}$ has $-\lambda_2 < -\lambda_1 < 0$.

To obtain exponential stability of this system by using an indefinite damping matrix, we consider the system $s' = \{\lambda_1, \lambda_2, \bar{g}, \bar{\delta}, d, -\alpha\}$ described in Eq. (A4), in which the first four elements of s' are the same as the corresponding elements in $\bar{r}$. Because $-\lambda_2 < -\lambda_1 < 0$, the first element of s' is now less than its second element, and the last two elements of s' are obtained by interchanging the last two element of s.

It is shown in Ref. [5] that the exponential stability region in the $\alpha - d$ plane for the system described by the ordered sextuple $s^* = \{\lambda_2, \lambda_1, \bar{g}, \bar{\delta}, -\alpha, d\}$ is the same as that for the system described by $s' = \{\lambda_1, \lambda_2, \bar{g}, \bar{\delta}, d, -\alpha\}$ [see Eqs. (A3) and (A4)] [5].

But in Case 1 a procedure is already given to get the explicit stability region of $s^* = \{\lambda_2, \lambda_1, \bar{g}, \bar{\delta}, -\alpha, d\}$ by using Eqs. (A5–A10), because the first element of the s^*, λ_2, is greater than its second element, λ_1.

Thus, when $-\lambda_2 < -\lambda_1 < 0$ in the system described by $-\lambda_2 < -\lambda_1 < 0$, we obtain the exponential stability region by 1) considering first the system $s^* = \{\lambda_2, \lambda_1, \bar{g}, \bar{\delta}, -\alpha, d\}$; 2) using Eqs. (A5–A10) to obtain the region of exponential stability of system s^* in the $\alpha - d$ plane; 3) picking any point $(\bar{\alpha}, \bar{d})$ that lies inside this region of exponential stability; and 4) using $\alpha = \bar{\alpha}$ and $d = \bar{d}$ in the system s'.

The system described by $\bar{s}' = \{\lambda_1, \lambda_2, \bar{g}, \bar{\delta}, \bar{d}, -\bar{\alpha}\}$ [or Eq. (A3)] with $\lambda_2 > \lambda_1 > 0$ is then guaranteed to be exponentially stable. Note that Eq. (A3) simply requires that the locations of $\bar{d}$ and $-\bar{\alpha}$ in the matrix D^I in Eq. (A11) be exchanged. Thus, for the gyroscopically stabilized system $\bar{r} = \{\lambda_1, \lambda_2, \bar{g}, \bar{\delta}\}$ with $-\lambda_2 < -\lambda_1 < 0$, the system

$$\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix} + \underbrace{\begin{bmatrix}-\bar{\alpha} & 0\\ 0 & \bar{d}\end{bmatrix}}_{D^I}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} + \underbrace{\begin{bmatrix}0 & \bar{g}\\ \bar{g} & 0\end{bmatrix}}_{\bar{G}}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} + \underbrace{\begin{bmatrix}-\lambda_1 & 0\\ 0 & -\lambda_2\end{bmatrix}}_{\Lambda}\begin{bmatrix}x_1\\ x_2\end{bmatrix},$$
$$-\lambda_2 < -\lambda_1 < 0,\ d, \alpha > 0 \tag{A12}$$

is exponentially stable for any $(\bar{\alpha}, \bar{d})$ belonging to the region of exponential stability in the $\alpha - d$ plane obtained by using Eqs. (A5–A10) for the system denoted $s^* = \{\lambda_2, \lambda_1, \bar{g}, \bar{\delta}, -\alpha, d\}$; the gyroscopically stabilized system $\bar{r} = \{\lambda_1, \lambda_2, \bar{g}, \bar{\delta}\}$ that has $-\lambda_2 < -\lambda_1 < 0$ is thereby rendered exponentially stable.

Case 3: This case is not used in the paper.

From Eqs. (A11) and (A12), dissipative damping d is provided to that degree of freedom for which the negative eigenvalue of the potential matrix is the smallest.

Appendix B: Necessary and Sufficient Condition for Pairing $2n$ Real Numbers in n Pairs so the Numbers in Each Individual Pair are Different

Result 3 deals with pairing $2n$ negative eigenvalues of the unstable potential matrix Λ into n pairs so that each pair has different eigenvalues in it. Here the more general problem of pairing a set of $2n$ real numbers to make n pairs so that each pair has different numbers in it is considered.

Consider $2n$ real numbers, n, of which k are distinct. Call these distinct numbers

$$a_1, a_2, \ldots a_k \tag{A13}$$

Let the multiplicity of $(a2)a_i$ be $m_i, i = 1, 2, \ldots, k$, so that $\sum_{p=1}^{k} m_p = 2n$. With no loss of generality, the set of these k distinct numbers can be ordered so that $m_1 \geq m_2 \geq \cdots \geq m_k$. Our aim is to create n pairs of numbers from these $2n$ numbers so that each pair has different numbers in it. Note that the number a_1 has multiplicity m_1, and this multiplicity m_1 is not exceeded by the multiplicities of any of the other distinct numbers $a_2, a_3, \ldots a_k$.

We next present three examples as a prelude to the results obtained below.

Example 1: $n = 5$. The $2n = 10$ numbers are, say, 5, 5, 5, 5, 5, 7, 7, 8, 8, 3.

Here, $a_1 = 5, a_2 = 7, a_3 = 8, a_4 = 3$; $m_1 = 5, m_2 = 2, m_3 = 2, m_4 = 1$. Note that $m_1 = n = 5$. A pairing of these numbers into 5 pairs, with each pair containing different numbers is [5, 7], [5, 7], [5, 8], [5, 8], [5, 3]. Such a pairing can be carried out in exactly $n - m_1 + 1 = 1$ (easy) step, because the number 5 is simply distributed among each of the 5 pairs. This pairing can be rewritten so that each pair has its smallest number as its first element: [5, 7], [5, 7], [5, 8], [5, 8], [3, 5] (see Result 2). The pairing is unique; no other pairing exists besides this one. □

Example 2: $n = 5$. The $2n = 10$ numbers are, say, 5, 5, 5, 7, 7, 7, 8, 8, 1, 3.

Here, $a_1 = 5, a_2 = 7, a_3 = 8, a_4 = 1, a_5 = 3$; $m_1 = 3, m_2 = 3, m_3 = 2, m_4 = 1, m_5 = 1$. Note that $m_1 < n$. A pairing of these numbers into 5 pairs, so that each pair contains different numbers is [1, 3], [7, 8], [5, 7], [5, 7], [5, 8]. Such a pairing can be obtained systematically by a process that takes exactly $n - m_1 + 1 = 3$ successive steps. The steps to get such a pairing are specified in the sufficiency proof of Result 1 below. Step 1 generates the pair [1, 3]; step 2, the pair [7, 8]; and, step 3 simultaneously generates the last three pairs.

There are, of course, other pairings, such as:
[5, 1], [8, 5], [3, 7], [5, 7], [8, 7]; and [1, 8], [3, 7], [5, 7], [5, 7], [5, 8].

As before, the first pairing above can be reorganized so that the first element of each pair is less that its second (see Result 2). □

Example 3: $n = 5$. The $2n = 10$ numbers are, say, 5, 5, 5, 5, 5, 5, 5, 8, 8, 3.

Here $a_1 = 7, a_2 = 8, a_3 = 3$; $m_1 = 7, m_2 = 2, m_3 = 1$, and $m_1 > n$. One cannot create a pairing of these numbers so that there are five pairs in which each pair contains different numbers (see Result A1 below). Thus, every pairing has one or more pairs that have identical numbers in them.

A pairing like [5, 8], [5, 8], [5, 3], [5, 5], [5, 5] gives the minimum number of pairs that have two identical numbers in them. In this case, this minimum number is 2. The procedure to get such a pairing is provided in Result A2. Note that here, $m_1 = 7 = n + r$, where $r = 2$. Result A2 in this Appendix says that in any pairing, the minimum number of pairs with identical numbers in them is r, which in this case is 2. □

Result A1: The necessary and sufficient condition for each of the n pairs formed from the $2n$ numbers described above to contain different numbers in them is that

$$m_1 \le n \tag{A14}$$

Proof:

1) Necessity: If a_1 has multiplicity $m_1 > n = n + r, 1 \le r \le n$, then $\sum_{p=2}^{k} m_p = n - r$. By first making pairs in which these $n - r$ numbers are each paired with, $n - r$ pairs that each have different numbers in them are created. This leaves $n + r - (n - r) = 2r$ remaining a_1's that need to be paired, and we therefore get r more pairs that each contain $[a_1, a_1]$. If instead, any pairs that contain different numbers are made from among the $n - r$ numbers for which $m_p, p > 1$, then there are more than $2ra_1$'s left for pairing, and again there would be at least one pair containing two a_1's.
Alternatively, if $m_1 > N$ then $\sum_{p=2}^{k} m_p < n$, and this would mean that one pair must contain two a_1's.

2) Sufficiency[†]: When $m_1 \le n$, a systematic process is developed of pairing the $2n$ numbers into N pairs each of which has different numbers in it. The proof gives a constructive way to produce such a pairing in $n - m_1 + 1$ steps.

When $m_1 = n$, place an a_1 in each of the n pairs. This yields the desired, unique pairing in a single step (Example 1 above).

When $m_1 < n$ (Example 2), then $\sum_{p=2}^{k} m_p > n$, and therefore there must be at least two distinct indices $i, j > 1$ with $m_i > 0$ and $m_j > 0$. Form a pair $[a_i, a_j]$. This is the first step in the process of generating the pairing. There are $n_1 = 2n - 2$ numbers at the end of this first step that now remain to be paired appropriately.

[†]An elegant alternative sufficiency proof by induction on n was given to the author by Professor Richard Arratia (personal communication with R. Arratia, Department of Mathematics, 406C Kaprielian Hall, Univ. of Southern California, Los Angeles). The present proof is constructive and shows that a pairing can be obtained in $n - m_1 + 1$ steps.

Since $m_1 \ge m_p, p > 1$, as long as the remaining numbers to be paired exceed $2m_1$, there are always distinct indices $i, j > 1$ with $m_i > 0$ and $m_j > 0$, so that a pair $[a_i, a_j]$ can be created from them. Hence, a series of such steps follow, each step contributing to a desirable pairing, with the number of numbers, n_s, left behind to be paired at the end of each successive step continually reducing by 2. At the end of $n - m_1$ such steps, $n - m_1$ pairs have been generated, each pair having different numbers in it. The number of numbers that remain to be paired now is $n_{n-m_1} = 2n - 2(n - m_1) = 2m_1$. This set of $2m_1$ numbers contains m_1 number of a_1's. A pairing in which an a_1 is placed in each of the m_1 pairs then simultaneously generates m_1 pairs each of which have different numbers in them. The pairing process therefore ends after $n - m_1 + 1$ steps, thus yielding a desired pairing of the $2n$ numbers. □

Thus, a set of $2n$ real numbers can be organized into n pairs such that the numbers in each pair are different if and only if the multiplicity of every number in the set does not exceed half the number of numbers, n, in the set. To the best of the author's knowledge this result is not available in this form in the mathematical literature.

Result A2: When $m_1 = n + r$, $1 \le r \le n$, the necessary and sufficient condition in Result A1 is not met, so there will be some pairs in every pairing that will contain the same numbers in them. For a given value of r, let the number of such pairs that contain identical numbers in them (in a pairing of the $2n$ numbers) be denoted by I_r. Then there is a pairing that gives the minimum value of I_r, which is r. A constructive proof that provides a systematic process to create such a pairing with $I_r = r$ is given.

Proof: Start the pairing process by first making pairs containing a_1 with each of the $n - r$ numbers that have multiplicities $m_p, p \ge 1$, thereby forming $n - r$ pairs each containing a_1. Each of these $n - r$ pairs has different numbers in it. This leaves behind a total of $n + r - (n - r) = 2r$ number of a_1's, i.e., r pairs that are each $[a_1, a_1]$. Any other pairing process that starts by first making any pairs $[a_i, a_j]$ that contain numbers with indices $i, j > 1$, has a larger number of a_1's left over, and therefore more than r pairs that are each $[a_1, a_1]$. Hence the minimum value of I_r equals r (Example 3). □

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